Cuspidalisations in Anabelian Geometry Week 6: Chain(Π)

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Our aim

"In this session, we aim to introduce the construction of Chain(Π) and related objects. We also want to go through a few examples and prove some basic results related to the theory of chains. We aim to cover Definition 4.2, Proposition 4.3 Example 4.4." – Yu Mao

Reference: [Moc12]

Shinichi Mochizuki. Topics in Absolute Anabelian Geometry I: Generalities, *J. Math. Sci. Univ. Tokyo*, 2012.

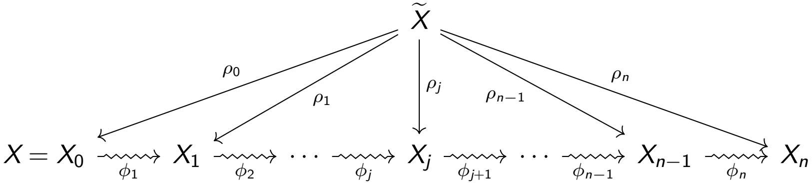
Definition 4.2 (i) and (ii)

Setup

- Let G be a slim profinite group.
  - G is slim means that every open subgroup H of G has trivial centraliser in G.
- Let
$$1 \to \Delta \to \Pi \to G \to 1$$
be an extension of GSAFG-type that admits base-prime partial construction data (k, X, Σ) with $\Sigma \neq \varnothing$ and X generically scheme-like.
  - k is a base field, $\Sigma \subseteq \mathfrak{Primes}$, and X/k is a generically scheme-like algebraic stack (it contains an open dense substack that is actually a scheme).
- Let $\alpha : \pi_1^{\mathrm{tame}}(X) \twoheadrightarrow \Pi$ be a scheme-theoretic envelope.
  - α factors like $\pi_1^{\mathrm{tame}}(X) \twoheadrightarrow \Pi^* \twoheadrightarrow \Pi$ where $1 \to \Delta \to \Pi \to G \to 1$ is isomorphic to an extension of the form $1 \to \Delta^* \to \Pi^*/N_{\Pi^*} \to G^*/N \to 1$.
- Let $\rho_0 : \widetilde{X} \to X$ be the pro-finite étale covering given by α.
- Write $\pi_1^{\mathrm{tame}}(X) \twoheadrightarrow G_k$ (absolute Galois group). Then α gives a scheme-theoretic envelope $\beta : G_k \twoheadrightarrow G$ (factors like $G_k \twoheadrightarrow G^* \twoheadrightarrow G$).
- Let $\widetilde{k}/k$ be the field extension given by β.

$\widetilde{X}/X$-chains

An $\widetilde{X}/X$-*chain of length* n is a sequence $t_1, \ldots, t_n \in \{\curlywedge, \curlyvee, \bullet, \circledcirc\}$, together with a commutative diagram of generically scheme-like algebraic stacks of the following form.



- ρ_0 is equal to the specified morphism $\widetilde{X} \to X$, "the $\widetilde{X}/X$".
- For $j > 0$, $\rho_j : \widetilde{X} \to X_j$ is a "rigidifying morphism".
- Each $\phi_j : X_j \rightsquigarrow X_{j+1}$ is an "elementary operation". This is a morphism $\phi_j : X_j \to X_{j+1}$ (or sometimes $X_{j+1} \to X_j$) of type t_j.

Rigidifying morphisms

For $j > 0$, $\rho_j : \widetilde{X} \to X_j$ is a dominant morphism with the following properties.

1. Firstly, there exists a morphism $X_j \to \mathrm{Spec}(k_j)$ such that $k_j/k \subseteq \widetilde{k}/k$ is a finite subextension of fields, X_j/k_j is geometrically connected, and the square below commutes.

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & \mathrm{Spec}(\widetilde{k}) \\ {\scriptstyle \rho_j}\downarrow & & \downarrow \\ X_j & \longrightarrow & \mathrm{Spec}(k_j) \end{array}$$

The morphism in (1) gives a maximal pro-finite étale covering $\widetilde{X}_j \to X_j$ through which ρ_j factors. This, in turn, induces a surjection

$$\mathrm{Gal}(\widetilde{X}_j/X_j) \twoheadrightarrow \mathrm{Gal}(\widetilde{k}/k_j).$$

2. Moreover, the kernel Δ_j of this surjection is slim and nontrivial.
3. Finally, if X/k is a hyperbolic orbicurve, then X_j/k_j is also a hyperbolic orbicurve and Δ_j is a pro-Σ group.

Elementary operations (the first two)

$t_j \in \{\curlywedge, \curlyvee, \bullet, \circledcirc\}$ determines the type of the elementary operation $\phi_j : X_j \rightsquigarrow X_{j+1}$.

- Type $\curlywedge$: finite étale covering.
  If $t_j = \curlywedge$, then $\phi_j : X_{j+1} \to X_j$ is a finite étale covering.
- Type $\curlyvee$: "finite étale quotient".
  If $t_j = \curlyvee$, then $\phi_j : X_j \to X_{j+1}$ is a finite étale morphism.

$\curlywedge$ is the only type of elementary operation $X_j \rightsquigarrow X_{j+1}$ for which the morphism goes the "wrong way", i.e. $X_{j+1} \to X_j$ instead of $X_j \to X_{j+1}$.

Previous conditions force the finite étale morphism $\phi_j : X_j \to X_{j+1}$ in type $\curlyvee$ to be a finite étale *cover*. This is because the composite $\widetilde{X} \to X_j \to X_{j+1}$ must equal $\widetilde{X} \to X_{j+1}$, and this last rigidifying morphism factors through the étale cover $\widetilde{X}_{j+1} \to X_{j+1}$.

The other types of elementary operations are allowed only in more restricted settings.

Elementary operations (the other two)

- Type $\bullet$: "de-cuspidalisation". This is only allowed when X/k is a hyperbolic orbicurve.
  If $t_j = \bullet$, then $\phi_j : X_j \hookrightarrow X_{j+1}$ is an open immersion whose image is the complement of a single k_{j+1}-rational point of X_{j+1} whose decomposition group in Δ_j is contained in some open normal torsion-free subgroup of Δ_j.
- Type $\circledcirc$: "de-orbification". This is only allowed when X/k is a hyperbolic orbicurve and $\Sigma = \mathfrak{Primes}$.
  If $t_j = \circledcirc$, then $\phi_j : X_j \to X_{j+1}$ is a partial coarsification morphism such that ϕ_j is an isomorphism over the complement in X_{j+1} of some k_{j+1}-rational point of X_{j+1}.
    - The coarse (moduli) space $C(X_j)$ of a hyperbolic orbicurve X_j is an algebraic space, together with a morphism of algebraic stacks $X_j \to C(X_j)$. It is characterised by a universal property (every morphism to an algebraic space factors uniquely through it), and it is functorial in X_j.
    - A dominant morphism of hyperbolic orbicurves $X_j \to X_{j+1}$ is a partial coarsification morphism if the induced morphism $C(X_j) \to C(X_{j+1})$ is an isomorphism.

Type-chains

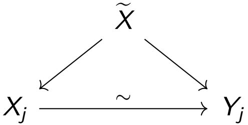
Given an $\widetilde{X}/X$-chain, we may forget the underlying commutative diagram of algebraic stacks, leaving us with a finite sequence of symbols in $\{\curlywedge, \curlyvee, \bullet, \circledcirc\}$. This is the *type-chain* associated to the $\widetilde{X}/X$-chain.

It should be noted that the commutative diagram part of the $\widetilde{X}/X$-chain data does not necessarily determine the type-chain. For example, if $\phi_j : X_j \to X_{j+1}$ is the identity morphism, then it may be an elementary operation of type $\curlywedge$ or of type $\curlyvee$.

There is exactly one $\widetilde{X}/X$-chain of length 0. It has empty type-chain and its commutative diagram is merely the specified morphism $\widetilde{X} \to X$, "the $\widetilde{X}/X$". It is fair to say that this is *the trivial $\widetilde{X}/X$-chain*.

Chain($\widetilde{X}/X$)

- Let $X_* : X_0 \rightsquigarrow \cdots \rightsquigarrow X_n$ and $Y_* : Y_0 \rightsquigarrow \cdots \rightsquigarrow Y_n$ be $\widetilde{X}/X$-chains with the same type-chain. An *isomorphism* $X_* \xrightarrow{\sim} Y_*$ is a collection of isomorphisms of generically-scheme like algebraic stacks $(X_j \xrightarrow{\sim} Y_j)$ that makes each of the following triangles commute.



  The diagonal arrows are the rigidifying morphisms.

- Chain($\widetilde{X}/X$) is the category whose objects are $\widetilde{X}/X$-chains and whose morphisms are the isomorphisms defined above (necessarily only between $\widetilde{X}/X$-chains of the same type-chain).

Note that Chain($\widetilde{X}/X$) is a groupoid.

$\mathrm{Chain}^{\mathrm{trm}}(\widetilde{X}/X)$

- Let $X_* : X_0 \rightsquigarrow \cdots \rightsquigarrow X_n$ and $Y_* : Y_0 \rightsquigarrow \cdots \rightsquigarrow Y_m$ be $\widetilde{X}/X$-chains with arbitrary type-chains. A *terminal morphism* $X_* \to Y_*$ is a dominant k-morphism $X_n \to Y_m$.
- $\mathrm{Chain}^{\mathrm{trm}}(\widetilde{X}/X)$ is the category whose objects are $\widetilde{X}/X$-chains and whose morphisms are terminal morphisms defined above (between $\widetilde{X}/X$-chains of arbitrary type-chain).

Terminal morphisms neglect almost all of the information in the chains they relate. Indeed, consider the category $\mathrm{Dom}(k)$ whose objects are generically scheme-like algebraic stacks over k and whose morphisms are dominant k-morphisms. Then the forgetful functor $\mathrm{Chain}^{\mathrm{trm}}(\widetilde{X}/X) \to \mathrm{Dom}(k) : X_* \mapsto X_n$ is fully faithful.

$\mathrm{Chain}^{\text{iso-trm}}(\widetilde{X}/X)$

- A *terminal isomorphism* between $\widetilde{X}/X$-chains with arbitrary type-chains is an isomorphism in the category $\mathrm{Chain}^{\text{trm}}(\widetilde{X}/X)$.
- $\mathrm{Chain}^{\text{iso-trm}}(\widetilde{X}/X)$ is the category whose objects are $\widetilde{X}/X$-chains and whose morphisms are terminal isomorphisms defined above (between $\widetilde{X}/X$-chains of arbitrary type-chain).

$\mathrm{Chain}^{\text{iso-trm}}(\widetilde{X}/X)$ is the largest subcategory of $\mathrm{Chain}^{\text{trm}}(\widetilde{X}/X)$ that is a groupoid.

- We have the functor $\mathrm{Chain}(\widetilde{X}/X) \to \mathrm{Chain}^{\text{iso-trm}}(\widetilde{X}/X)$ that is the identity on objects and forgetful on morphisms.
- We have the functor $\mathrm{Chain}^{\text{iso-trm}}(\widetilde{X}/X) \to \mathrm{Chain}^{\text{trm}}(\widetilde{X}/X)$ that is the identity on objects and the inclusion on morphisms.

Terminal isomorphisms

Proposition 4.3 and Example 4.4 will only use terminal isomorphisms, i.e. we'll work in the category $\mathrm{Chain}^{\text{iso-trm}}(\widetilde{X}/X)$.

Let (X) be the trivial $\widetilde{X}/X$-chain. Given an $\widetilde{X}/X$-chain $X_* : X_0 \rightsquigarrow \cdots \rightsquigarrow X_n$, there is a terminal isomorphism $X_* \to (X)$ if and only if there is a k-isomorphism $X_n \xrightarrow{\sim} X$. We regard this as a trivial situation because elementary operations ought to take us from $X = X_0$ to a "simpler" X_n.

Proposition 4.3

The statement

Proposition 4.3 (Re-ordering of chains)

Assume the notation of Definition 4.2. Suppose that $\Sigma = \mathfrak{Primes}$. Then, for any $\widetilde{X}/X$-chain $X_0 \rightsquigarrow \cdots \rightsquigarrow X_n$, there exists a terminally isomorphic $\widetilde{X}/X$-chain $Y_0 \rightsquigarrow \cdots \rightsquigarrow Y_m$ whose type-chain has the form

$$\curlywedge, \bullet, \cdots, \bullet, \curlyvee, \circledcirc, \cdots, \circledcirc, \curlyvee.$$

Moreover, we may force the last finite étale morphism $Y_{m-1} \to Y_m$ (type $\curlyvee$) to arise from a finite extension of the base field.

- Note that $X_0 = X = Y_0$ by the definition of an $\widetilde{X}/X$-chain.
- Moreover, $X_n \simeq_k Y_m$ by the definition of terminally isomorphic.
- Since $\Sigma = \mathfrak{Primes}$, the base field k is of characteristic zero.

Simple case of the proof

First, we suppose that X is *not* a hyperbolic orbicurve.
So, $X_* : X_0 \rightsquigarrow \cdots \rightsquigarrow X_n$ has type-chain $t_1, \ldots, t_n \in \{\curlywedge, \curlyvee\}$.

We will perform the following steps.

1. Move $\curlywedge$s to the left
2. Compose $\curlyvee$s or $\curlywedge$s
3. Possibly extend by the identity with type $\curlyvee$ (given by trivial base change)

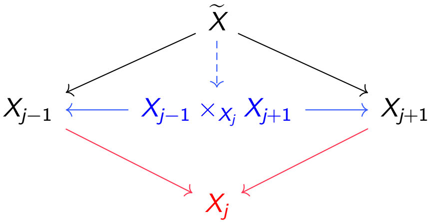
So the form of the type-chain will change in the manner below

$$t_1, \ldots, t_n \qquad \curlywedge, \cdots, \curlywedge, \curlyvee, \cdots, \curlyvee \qquad \curlywedge, \curlyvee \qquad \curlywedge, \curlyvee, \curlyvee.$$

Moving $\curlywedge$s to the left

Suppose that $t_j, t_{j+1} = \curlyvee, \curlywedge$. Then we may swap these types by replacing $X_{j-1} \rightsquigarrow X_j \rightsquigarrow X_{j+1}$ with $X_{j-1} \rightsquigarrow X_{j-1} \times_{X_j} X_{j+1} \rightsquigarrow X_{j+1}$.

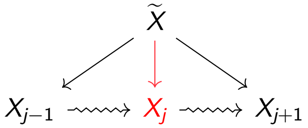
- Applying $X_{j-1} \times_{X_j} -$ to the finite étale cover $X_{j+1} \to X_j$ gives us another finite étale cover $X_{j-1} \times_{X_j} X_{j+1} \to X_{j-1}$.
- Applying $- \times_{X_j} X_{j+1}$ to our finite étale morphism $X_{j-1} \to X_j$ gives us another finite étale morphism $X_{j-1} \times_{X_j} X_{j+1} \to X_{j+1}$.
- The rigidifying morphism $\widetilde{X} \to X_{j-1} \times_{X_j} X_{j+1}$ is the unique morphism afforded to us by the universal property of the fibre product.



Composing $\curlywedge$s or $\curlyvee$s

Suppose that $t_j = t_{j+1} \in \{\curlywedge, \curlyvee\}$. Then the elementary operations $X_{j-1} \rightsquigarrow X_j$ and $X_j \rightsquigarrow X_{j+1}$ may be composed to form a single elementary operation $X_{j-1} \rightsquigarrow X_{j+1}$ of the same type.

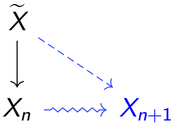
- The composite of two finite étale covers is a finite étale cover, and the composite of two finite étale morphisms is a finite étale morphism.
- The rigidifying morphisms $\widetilde{X} \to X_{j-1}$ and $\widetilde{X} \to X_{j+1}$ form a commutative triangle with $X_{j-1} \rightsquigarrow X_{j+1}$.



Extending by an isomorphism of type $\curlywedge$ or $\curlyvee$

We may extend X_* by a k-isomorphism $X_n \rightsquigarrow X_{n+1}$, assigning it type $\curlywedge$ or $\curlyvee$. If we want type $\curlywedge$, then we make the isomorphism go this way: $X_n \xrightarrow{\sim} X_{n+1}$. If we want type $\curlyvee$, then we make the isomorphism go the other way: $X_{n+1} \xrightarrow{\sim} X_n$.

- An isomorphism is both a finite étale covering (type $\curlywedge$) and a finite étale morphism (type $\curlyvee$).
- The rigidifying morphism $\widetilde{X} \to X_{n+1}$ is the unique morphism that makes the following triangle commute.



This completes the proof in the simple case.

Harder case of the proof

We now assume that X/k is a hyperbolic orbicurve (and $\Sigma = \mathfrak{Primes}$).
So, $X_* : X_0 \rightsquigarrow \cdots \rightsquigarrow X_n$ has arbitrary type-chain $t_1, \ldots, t_n$.

We perform the following steps (like before).

1. Move $\curlywedge$s to the left
2. Compose $\curlywedge$s

The result is a new chain $X'_* : X_0 \rightsquigarrow X'_1 \rightsquigarrow \cdots \rightsquigarrow X'_{n'} = X_n$ with type-chain $\curlywedge, t'_2, \ldots, t'_{n'}$ where each $t'_j \in \{\Upsilon, \bullet, \circledcirc\}$.

We may compose the morphisms underlying $X'_1 \rightsquigarrow \cdots \rightsquigarrow X'_{n'}$ to get $X'_1 \to X'_{n'}$. We want to modify X'_* to get a new chain $X''_* : X_0 \rightsquigarrow X''_1 \rightsquigarrow \cdots \rightsquigarrow X''_{n''} \rightsquigarrow X''_{n''+1} = X_n$ with type-chain $\curlywedge, t''_2, \ldots, t''_{n''}, \Upsilon$ where each $t''_j \in \{\Upsilon, \bullet, \circledcirc\}$. This is so that the composite morphism $X''_1 \to X''_{n''}$ has some nice properties.

Our final chain $Y_* : Y_0 \rightsquigarrow Y_1 \rightsquigarrow \cdots \rightsquigarrow Y_{m-1} \rightsquigarrow Y_m$ will look like

- $Y_0 \rightsquigarrow Y_1 = X''_0 \rightsquigarrow X''_1$
- $Y_{m-1} \rightsquigarrow Y_m = X''_{n''} \rightsquigarrow X''_{n''+1}$
- $Y_1 \rightsquigarrow \cdots \rightsquigarrow Y_{m-1}$ may look quite different from $X''_1 \rightsquigarrow \cdots \rightsquigarrow X''_{n''}$

Refining the composite morphism

We want the morphism $X''_1 \to X''_{n''}$ to have the following properties.

- X''_1 is a hyperbolic curve
- The morphism induces a Galois extension of function fields
- It also induces an isomorphism of base fields
- $X''_1 \to X''_{n''}$ factors through a connected finite étale covering $Z \to X''_{n''}$ with Z a hyperbolic curve

With this goal in mind, we may use the following tricks.

- Take an étale cover $X''_1 \to X'_1$ to replace $X_0 \rightsquigarrow X'_1$ with $X_0 \rightsquigarrow X''_1 \rightsquigarrow X'_1$
  - $X_0 \rightsquigarrow X''_1 \rightsquigarrow X'_1$ looks like $X_0 \leftarrow X'_1 \leftarrow X''_1 \to X'_1$
- Extend X'_* with a base change $X''_{n''} \rightsquigarrow X'_{n'} = X_n$, base changing operations to the left as necessary
  - Base changing étale covers of type $\curlyvee$ is fine. Base changing $\circledcirc$ and $\bullet$ is fine because their end points have the same base field. Eventually, we'll get to the étale cover of type $\curlywedge$, which can "absorb" the base change.
- Splice in a connected finite étale covering $Z \to X''_{n''}$ via $X''_{n''} \rightsquigarrow Z \rightsquigarrow X''_{n''}$ and move the $\curlyvee$ (on the left) all the way to the left.

Elementary operations from the function field

Now, with this nice composite morphism $X''_1 \to X''_{n''}$ in hand, we can replace the elementary operations $X''_1 \rightsquigarrow X''_{n''}$ with our nicer $Y_1 \rightsquigarrow Y_{m-1}$.

- To each cusp of X''_1 that maps to a point of $X''_{n''}$, we may apply a de-cuspidalisation operation. We repeat this finitely many times until we end up with a "de-cusped" hyperbolic curve.
- We take the stack quotient of this new curve by $\mathrm{Gal}(X''_1/X''_{n''})$, an operation of type Υ, to get a hyperbolic orbicurve.
- To finitely many stacky points of this orbicurve, we apply de-orbification operations until we reach $X''_{n''}$.

This part of the chain has type-chain of the desired form $\bullet, \cdots, \bullet, \Upsilon, \circledcirc, \cdots, \circledcirc$.

Example 4.4

Non-permutability of étale quotients and de-orbifications

Assume the notation of Definition 4.2. Suppose that $\Sigma = \mathfrak{Primes}$.

First, we will show that there exists an

- $\widetilde{X}/X$-chain $X_0 \rightsquigarrow X_1 \rightsquigarrow X_2$ with type-chain $\curlyvee, \circledcirc$

that is not terminally isomorphic to any

- $\widetilde{X}/X$-chain $Y_0 \rightsquigarrow Y_1 \rightsquigarrow Y_2$ with type-chain $\circledcirc, \curlyvee$.

Then we will show that the same thing holds with the type-chains reversed.

$\Upsilon, \circledcirc$ cannot always be transformed into $\circledcirc, \Upsilon$

Let X/k be a hyperbolic curve, together with an automorphism $\sigma \in \mathrm{Aut}_k(X)$ of order 2 admitting exactly one fixed point x. We assume that $x \in X(k)$.

- The quotient stack morphism $X_0 := X \to [X/\langle\sigma\rangle] =: X_1$ is a finite étale covering and gives an elementary operation $X_0 \rightsquigarrow X_1$ of type Υ. Let $x_1 \in X_1(k)$ be the image of $x \in X_0(k)$, noting that we have $k_1 = k$.
- We have an elementary operation $X_1 \rightsquigarrow X_2$ of type $\circledcirc$ with X_2 a hyperbolic curve. Moreover, the underlying partial coarsification morphism $X_1 \to X_2$ is an isomorphism over $X_2 - \{x_2\}$, where $x_2 \in X_2(k)$ is the image of $x_1 \in X_1$. This is "essentially unique".

Let $(g, r), (g_2, r_2)$ be the types of the hyperbolic curves X, X_2. Write $\chi = 2g - 2 + r > 0$ and $\chi_2 = 2g_2 - 2 + r_2 > 0$. The composite morphism $X = X_0 \to X_1 \to X_2$ is a ramified covering of degree 2. So, the Riemann-Hurwitz formula tells us that

$$\chi = 2\chi_2 + 1.$$

Arriving at a contradiction

Suppose that $Y_0 \rightsquigarrow Y_1 \rightsquigarrow Y_2$ is an $\widetilde{X}/X$-chain of type $\circledcirc, \curlyvee$ that is terminally isomorphic to X_*.

Since $X = Y_0$ is a scheme, the de-orbification $Y_0 \rightsquigarrow Y_1$ is given by an isomorphism. Since Y_* is terminally isomorphic to X_*, we have a k-isomorphism $Y_2 \xrightarrow{\sim} X_2$. The composite morphism $X = Y_0 \xrightarrow{\sim} Y_1 \rightarrow Y_2 \xrightarrow{\sim} X_2$ is an unramified covering of some finite degree d. So, the Riemann-Hurwitz formula tells us that

$$\chi = d\chi_2.$$

Comparing this with our previous formula for χ, we have $2\chi_2 + 1 = d\chi_2$. We then rearrange this to $1 = (d-2)\chi_2$. Since d and χ_2 are both positive integers, this forces $d - 2 = 1 = \chi_2$. So we have $d = 3$ and $\chi_2 = 1$, and hence also $\chi = 3$.

If our initial hyperbolic curve X has $\chi > 3$, then we arrive at a contradiction. Note that any hyperbolic curve X of type (g, r) with $g \geq 3$ necessarily has $\chi = 2g - 2 + r > 3$.

$\circledcirc, \Upsilon$ cannot always be transformed into $\Upsilon, \circledcirc$

Let X/k be a proper hyperbolic orbicurve. Let $X \to C$ be the coarse space of X. We assume that the following three conditions are met.

- C is a proper hyperbolic curve over k
- $X \to C$ is not an isomorphism, but it restricts to an isomorphism away from some k-rational point c of C
- there exists a finite étale covering $\epsilon : C \to D$ of degree 2.

Then D is a proper hyperbolic curve over k that is not isomorphic to C. We may cook up examples by starting with D.

- $X \to C$ gives us an elementary operation $X_0 := X \rightsquigarrow C := X_1$ of type $\circledcirc$.
- $\epsilon : C \to D$ gives us an elementary operation $X_1 := C \rightsquigarrow D := X_2$ of type Υ.

Let $e_x \geq 2$ be the ramification index of $X \to C$ at the unique $x \in X(k)$ lying over $c \in C(k)$. Let $g_D \geq 2$ be the genus of D, and let $\chi_D = 2g_D - 2 \geq 2$.

Almost arriving at a contradiction

Suppose that $Y_0 \rightsquigarrow Y_1 \rightsquigarrow Y_2$ is an $\widetilde{X}/X$-chain of type $\Upsilon, \circledcirc$ that is terminally isomorphic to X_*.

- We have a de-orbification morphism $Y_1 \to Y_2$ from a hyperbolic orbicurve Y_1 to a hyperbolic curve $Y_2 \simeq X_2 = D$. Therefore, Y_1 is a scheme away from some $y_1 \in Y_1(k)$. Note that Y_1 is not a scheme at y_1, for otherwise the finite étale cover $Y_0 = X$ would also be a scheme. Therefore, the ramification index e_{y_1} of $Y_1 \to Y_2$ at y_1 satisfies $e_{y_1} \geq 2$.
- Suppose, for a contradiction, that the finite étale cover $Y_0 \to Y_1$ is not an isomorphism. Then it is of degree $d \geq 2$. We have two composite coverings, $X \to C \to D$ and $X = Y_0 \to Y_1 \to Y_2 \xrightarrow{\sim} D$. The Riemann-Hurwitz formula gives

$$2\chi_D + \frac{e_x - 1}{e_x} = d\left(\chi_D + \frac{e_{y_1} - 1}{e_{y_1}}\right).$$

We get the contradiction (use $\chi_D \geq 0$, $d \geq 2$ and $\frac{1}{2} \leq \frac{t-1}{t} < 1$ for $t \geq 2$)

$$1 > \frac{e_x - 1}{e_x} = (d-2)\chi_D + d\frac{e_{y_1} - 1}{e_{y_1}} \geq d\frac{e_{y_1} - 1}{e_{y_1}} \geq d\frac{1}{2} \geq 1.$$

Finally arriving at a contradiction

- So the finite étale cover $Y_0 \to Y_1$ is an isomorphism, $X = Y_0 \xrightarrow{\sim} Y_1$. Composing this with the de-orbification $Y_1 \rightsquigarrow Y_2$, we get a de-orbification $X \rightsquigarrow Y_2$ that is not an isomorphism.
- Recall that the coarse space morphism $X \to C$ is not an isomorphism, but it restricts to an isomorphism away from some k-rational point c of C.
- It follows that $Y_2 \simeq C$, the coarse space of X.
- But then this gives us the contradiction $D = X_2 \simeq Y_2 \simeq C$.

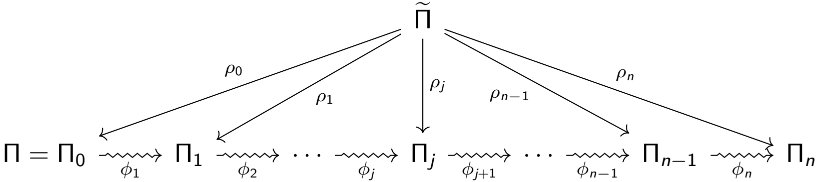
Definition 4.2 (iii) and (iv)

Technical matter

- Let $\mathcal{U}$ be the set of open subgroups of Π. We define a relation $\leq$ on $\mathcal{U}$ by reverse inclusion, i.e. $U \leq V \iff U \supseteq V$. This makes $(\mathcal{U}, \leq)$ a directed system.
- If $U, V \in \mathcal{U}$ with $U \leq V$, then we have the inclusion morphism $\iota_{UV} : V \hookrightarrow U$. This defines an inverse system $\widetilde{\Pi}$ of topological groups over $(\mathcal{U}, \leq)$.
- Suppose that Π' is another profinite group. Then we may form an inverse system (Π') of topological groups over $(\mathcal{U}, \leq)$ with the identity morphism on Π' taken for each $U \leq V$.
- A morphism $\widetilde{\Pi} \to (\Pi')$ of these inverse systems of topological groups consists of a collection of topological group morphisms $(\varphi_U : U \to \Pi')_{U \in \mathcal{U}}$ such that if $U \leq V$, then $\varphi_V = \varphi_U|_V$.
- We will consider morphisms $\widetilde{\Pi} \to (\Pi')$ only in a restricted setting. The profinite groups Π, Π' will always be slim, and the constituent morphisms φ_U will always be open. As such, a morphism $\widetilde{\Pi} \to (\Pi')$ will be the same as a pair (U, φ_U) consisting of a particular $U \in \mathcal{U}$ and an open topological group morphism $\varphi_U : U \to \Pi'$.

Π-chain

A Π-*chain of length n* is a sequence $t_1, \ldots, t_n \in \{\curlywedge, \Upsilon, \bullet, \circledcirc\}$, together with a commutative diagram of inverse systems of topological groups of the following form.



- ρ_0 is equal to the trivial morphism $\widetilde{\Pi} \to \Pi$.
- For $j > 0$, $\rho_j : \widetilde{\Pi} \to \Pi_j$ is a "rigidifying morphism".
- Each $\phi_j : \Pi_j \rightsquigarrow \Pi_{j+1}$ is an "elementary operation". This is a morphism $\phi_j : \Pi_j \to \Pi_{j+1}$ (or sometimes $\Pi_{j+1} \to \Pi_j$) of type t_j.

The inverse systems along the bottom row are all constant, so the morphisms between them are really just topological group morphisms.

Rigidifying morphisms

For $j > 0$, $\rho_j : \widetilde{\Pi} \to \Pi_j$ is an open morphism with the following properties.

1. Firstly, there exists a surjection $\Pi_j \twoheadrightarrow G_j$ onto an open subgroup $G_j \subseteq G$ such that the square below commutes.

$$\begin{array}{ccccc} \widetilde{\Pi} & \longrightarrow & \Pi & \twoheadrightarrow & G \\ \downarrow & & & & \uparrow \\ \Pi_j & & \twoheadrightarrow & & G_j \end{array}$$

2. The kernel Δ_j of $\Pi_j \twoheadrightarrow G_j \hookrightarrow G$ is slim and nontrivial. Moreover, if p is a prime number that divides the order of some finite quotient group of Δ_j, then p is invertible in k.
3. Finally, if X/k is a hyperbolic orbicurve, then Δ_j is a pro-Σ group.

Elementary operations (the first two)

$t_j \in \{\curlywedge, \curlyvee, \bullet, \circledcirc\}$ determines the type of the elementary operation $\phi_j : X_j \rightsquigarrow X_{j+1}$.

- Type $\curlywedge$:
  If $t_j = \curlywedge$, then $\phi_j : \Pi_{j+1} \to \Pi_j$ is an open immersion of profinite groups.
- Type $\curlyvee$:
  If $t_j = \curlyvee$, then $\phi_j : \Pi_j \to \Pi_{j+1}$ is an open immersion of profinite groups.

As before, $\curlywedge$ is the only type of elementary operation $\Pi_j \rightsquigarrow \Pi_{j+1}$ for which the morphism goes the "wrong way", i.e. $\Pi_{j+1} \to \Pi_j$ instead of $\Pi_j \to \Pi_{j+1}$.

The other types of elementary operations are allowed only in more restricted settings.

Elementary operations (the other two)

- Type $\bullet$: "de-cuspidalisation". This is only allowed when X/k is a hyperbolic orbicurve.
  If $t_j = \bullet$, then $\phi_j : \Pi_j \twoheadrightarrow \Pi_{j+1}$ is a surjection of profinite groups whose kernel is topologically normally generated by a *cuspidal decomposition group C in* Δ_j, with the additional constraint that this C is contained in some normal open torsion-free subgroup of Δ_j.
  - Recall the subgroup $\Delta \subseteq \Pi$ determined by the scheme-theoretic envelope α. Then there exists a cusp x of X, with decomposition group Δ_x in Δ, such that $\rho_j(\rho_0^{-1}(\Delta_x))$ is a nontrivial subgroup of Δ_j. Moreover, we may choose this x so that C is the commensurator of $\rho_j(\rho_0^{-1}(\Delta_x))$ in Δ_j.
- Type $\circledcirc$: "de-orbification". This is only allowed when X/k is a hyperbolic orbicurve and $\Sigma = \mathfrak{Primes}$.
  If $t_j = \circledcirc$, then $\phi_j : X_j \twoheadrightarrow X_{j+1}$ is a surjection of profinite groups whose kernel is topologically normally generated by a finite closed subgroup of Δ_j.

- As with $\widetilde{X}/X$-chain, we may consider the *type-chain* associated to a Π-chain, by simply forgetting the diagram part of the data.
- Let Π_* and Ψ_* be two Π-chains with the same type-chain. An *isomorphism* $\Pi_* \xrightarrow{\sim} \Psi_*$ is a collection of isomorphisms of profinite groups $(\Pi_j \xrightarrow{\sim} \Psi_j)$ that makes each of the following triangles commute.

$$\begin{array}{ccc} & \widetilde{\Pi} & \\ \swarrow & & \searrow \\ \Pi_j & \xrightarrow{\sim} & \Psi_j \end{array}$$

- Chain(Π) is the category of Π-chains whose morphisms are the isomorphisms defined above. Again, note that these morphisms are only defined between Π-chains with the same type-chain.

Terminal considerations

- Let $\Pi_* : \Pi_0 \rightsquigarrow \cdots \rightsquigarrow \Pi_n$ and $\Psi_* : \Psi_0 \rightsquigarrow \cdots \rightsquigarrow \Psi_m$ be Π-chains with arbitrary type chains. A *terminal homomorphism* $\Pi_* \to \Psi_*$ is an open homomorphism $\Pi_n \to \Psi_m$ forming a commutative diagram

$$\begin{array}{ccccc} \Pi_n & & \longrightarrow & & \Psi_n \\ {\scriptstyle\sim}\downarrow & & & & \downarrow{\scriptstyle\sim} \\ \Pi_n & & & & \Psi_n \\ & \searrow & & \swarrow & \\ & & G & & \end{array}$$

  with some inner-automorphisms (the vertical arrrows).
- $\mathrm{Chain}^{\mathrm{trm}}(\Pi)$ is the category of Π-chains whose morphisms are the terminal homomorphisms defined above. The isomorphisms in $\mathrm{Chain}^{\mathrm{trm}}(\Pi)$ are *terminal isomorphisms*. $\mathrm{Chain}^{\mathrm{iso\text{-}trm}}(\Pi)$ is the category of Π-chains whose morphisms are terminal isomorphisms.
- We have functors $\mathrm{Chain}(\Pi) \to \mathrm{Chain}^{\mathrm{iso\text{-}trm}}(\Pi) \hookrightarrow \mathrm{Chain}^{\mathrm{trm}}(\Pi)$.

Compatibility of $\widetilde{X}/X$-chains with Π-chains

Assume the notation of Definition 4.2.

- Let $X_* : X_0 \rightsquigarrow \cdots \rightsquigarrow X_n$ be an $\widetilde{X}/X$-chain with type-chain $t_1, \ldots, t_n$.
- For each j, let $\Pi_j = \mathrm{Gal}(\widetilde{X}_j/X_j)$ where $\widetilde{X}_j \to X_j$ is the maximal pro-finite étale covering induced by the rigidifying morphism $\widetilde{X} \to X_j$.
- Then $\Pi_* : \Pi_0 \rightsquigarrow \cdots \rightsquigarrow \Pi_n$ is a Π-chain with type-chain $t_1, \ldots, t_n$.
- $X_* \mapsto \Pi_*$ defines a functor $\mathrm{Chain}(\widetilde{X}/X) \to \mathrm{Chain}(\Pi)$.
- We similarly have functors $\mathrm{Chain}^{\mathrm{trm}}(\widetilde{X}/X) \to \mathrm{Chain}^{\mathrm{trm}}(\Pi)$ and $\mathrm{Chain}^{\text{iso-trm}}(\widetilde{X}/X) \to \mathrm{Chain}^{\text{iso-trm}}(\Pi)$.