Cuspidalisations in Anabelian Geometry

Week 10: Reconstruction of Curves

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2025

Additive Structures

Let X be a hyperbolic curve defined over a field k.

Proposition 1

Suppose that X is proper, k is algebraically closed. Let D be a divisor on X, we write $\Gamma^\times(D) := \{f \in K(X)^\times : \mathrm{div}(f) + D \text{ is effective}\}$. Moreover, we write $l(D) := \dim_k H^0(X, \mathcal{O}_X(D))$. Then the followings hold:

(i) Suppose that $\Gamma^\times(D)$ is non-empty, then $k^\times$ acts freely on $\Gamma^\times(D)$.

(ii) The integer $l(D) \geq 0$ is equal to the smallest non-negative integer d such that there exists an effective divisor E of degree d on X for which $\Gamma^\times(D - E) = \emptyset$.

Proposition 2

We keep the notions in Proposition 1. Then the followings hold:
(i) There exists distinct points $x, y_1, y_2 \in X(k)$, together with a divisor D on X such that $x, y_1, y_2 \notin \mathrm{Supp}(D)$, such that $l(D) = 2$ and $l(D - E) = 0$, for any effective divisor $E = e_1 + e_2$, where $e_1 \neq e_2$, $\{e_1, e_2\} \subset \{x, y_1, y_2\}$.
(ii) We keep the notions in (i). Then for $i = 1, 2$, $\lambda \in k^\times$, there exists a unique element $f_{\lambda,i} \in \Gamma^\times(D)$ such that $f_{\lambda,i}(x) = \lambda$, $f_{\lambda,i}(y_i) \neq 0$ and $f_{\lambda,i}(y_{3-i}) = 0$.
(iii) We keep the notions in (ii). Let $\lambda, \mu \in k^\times$ such that $\lambda/\mu \neq -1$. Let $f_{\lambda,1} \in \Gamma^\times(D)$ and $f_{\mu,2} \in \Gamma^\times(D)$. Then $f_{\lambda,1} + f_{\mu,2} \in \Gamma^\times(D)$ can be characterised as the unique element $g \in \Gamma^\times(D)$ such that $g(y_1) = f_{\lambda,1}(y_1)$, $g(y_2) = f_{\mu,2}(y_2)$. In particular, $\lambda + \mu \in k^\times$ can be characterised as $g(x) \in k^\times$.

Proof of Proposition 2

Let D be a divisor on X with $l(D) \geq 2$. By possibly replacing D by $D - E$ for some E, we may assume that $l(D) = 2$.

- Take $x \in X(k) \setminus \mathrm{Supp}(D)$ such that $\mathcal{O}_X(D)$ admits a global section that does not vanish at x.
- Take $y_1 \in X(k) \setminus (\mathrm{Supp(D)} \cup \{x\})$ to be any point such that $\mathcal{O}_X(D - x)$ admits a global section that does not vanish at y_1.
- Take $y_2 \in X(k) \setminus (\mathrm{Supp(D)} \cup \{x, y_1\})$ to be any point such that $\mathcal{O}_X(D - x)$ and $\mathcal{O}_X(D - y_1)$ admit global sections that do not vanish at y_2.

One checks immediately that x, y_1, y_2 satisfy the conditions in assertion (i). Assertions (ii),(iii) are consequences of assertion (i). This proves Proposition 2.

Additive Structures

Proposition 3

We keep the notions in Proposition 2. Then the additive structure of $k(X)^\times \cup \{0\}$ can be reconstructed from the following data:

- The multiplicative group $k(X)^\times$.
- The set of surjections arising from valuations at points $x \in X(k)$:

$$\mathcal{V}_X := \{\mathrm{ord}_x : k(X)^\times \twoheadrightarrow \mathbb{Z}\}.$$

- For each $v := \mathrm{ord}_x \in \mathcal{V}_X$, the subgroup $\mathcal{U}_v := \{f \in k(X)^\times : f(x) = 1\}$ of $k(X)^\times$.

Additive Structures

More generally. Let $(M, S, \{U_s\}_{s \in S})$ be a triple such that

- M is an abelian group, together with an isomorphism $\alpha : M \xrightarrow{\sim} k(X)^\times$.
- $S := \{s : M \twoheadrightarrow \mathbb{Z}\}$ a set of homomorphisms $s : M \twoheadrightarrow \mathbb{Z}$ such that for each $f \in M$, only finitely many $s \in S$ with $s(f) \neq 0$, together with a bijection $g : S \xrightarrow{\sim} \mathcal{V}_X$.
- For each $s \in S$, $U_s \subset M$ is defined to be $\ker(s)/\bigcap_{s \in S} \ker(s)$.

Then there is a reconstruction of a field M^{fld}, such that α extends uniquely to an isomorphism of fields

$$\alpha^+ : M^{\mathrm{fld}} \xrightarrow{\sim} k(X).$$

Proof of Proposition 3

Step 1: We write $\mathcal{C} := \bigcap_{s \in S} \ker(s)$, which corresponds to the multiplicative group of constant function field $k^{\times}$. Moreover, we have $\ker(s) = U_s \times \mathcal{C}$.
Step 2: We write $\mathcal{D}iv(S)$ for the free abelian group on S. Moreover, for each $f \in M$, we shall write

$$\operatorname{div}(f) := \sum_{s \in S} s(f)[s] \in \mathcal{D}iv(S).$$

Hence we can define

$$\Gamma^{\times}(\mathcal{D}) := \{f \in M : \operatorname{div}(f) + \mathcal{D} \geq 0\}$$

where $\mathcal{D} \in \mathcal{D}iv(S)$ (an element $\mathcal{D}$ in $\mathcal{D}iv(S)$ is said to be effective denoted by $\mathcal{D} \geq 0$ if all coefficient of $\mathcal{D}$ is non-negative).

Proof of Proposition 3 (continued)

Step 3: Let $\mathcal{D} \in \mathcal{D}iv(S)$. We associate an integer $\ell(\mathcal{D})$ to $\mathcal{D}$, defined as the smallest non-negative integer d such that there exists an element $\mathcal{E} \in \mathcal{D}iv(S)$ of degree d for which $\Gamma^{\times}(\mathcal{D} - \mathcal{E}) = \emptyset$ (here the degree of an element of $\mathcal{E} \in \mathcal{D}iv(S)$ is defined to be the sum of all coefficients of $\mathcal{E}$).
Step 4: We apply Proposition 2 (ii) (iii), to recover a field structure of $\mathcal{C}^{\mathsf{fld}} := \mathcal{C} \cup \{0\}$, hence also the field structure of M^{fld}.

Cyclotomic Synchronisation

Definition 4

Let X be a proper hyperbolic curve over a field k of characteristic 0. The **geometric cyclotome** of X is defined to be

$$M_X := \mathrm{Hom}(H^2(\Delta_X, \widehat{\mathbb{Z}}), \widehat{\mathbb{Z}}).$$

Lemma 5

The geometric cyclotome M_X is canonically isomorphic to $\widehat{\mathbb{Z}}(1)$ as a G_k-module.

Proof of Lemma 5

Recall that $H^2(\Delta_X, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\sim} H^2_{\text{ét}}(X_{\bar{k}}, \underline{\mathbb{Z}/n\mathbb{Z}})$. Then by Poincare duality, we have a perfect pairing

$$H^2_{\text{ét}}(X_{\bar{k}}, \underline{\mathbb{Z}/n\mathbb{Z}}) \times H^0_{\text{ét}}(X_{\bar{k}}, \mu_n) \to H^0_{\text{ét}}(X_{\bar{k}}, \mu_n)$$

hence there exists a natural isomorphism

$$H^0_{\text{ét}}(X_{\bar{k}}, \mu_n) \xrightarrow{\sim} H^2_{\text{ét}}(X_{\bar{k}}, \underline{\mathbb{Z}/n\mathbb{Z}})^\vee = \text{Hom}(H^2_{\text{ét}}(X_{\bar{k}}, \underline{\mathbb{Z}/n\mathbb{Z}}), \mathbb{Z}/n\mathbb{Z}).$$

On the other hand, $H^0_{\text{ét}}(X_{\bar{k}}, \mu_n) \xrightarrow{\sim} \mu_n$. By taking projective limit, Lemma 5 follows.

Proposition 6

Let X be a proper hyperbolic curve over a field k of characteristic 0. Let $U \subset X$ be a non-empty open subcurve. Then for each $x \in X(k) \setminus U(x)$, the cuspidal inertia subgroup I_x is canonically isomorphic to M_X.

Proof of Proposition 6

We write $\Delta_{U_x}^{\text{c-cn}}$ for the maximal cuspidally central quotient of of Δ_{U_x}, where $U_x := X \setminus \{x\}$. Hence we have an exact sequence

$$1 \to I_x \to \Delta_{U_x}^{\text{c-cn}} \to \Delta_X \to 1.$$

Then we apply the Hoschild-Serre spectral sequence to this sequence, we have a map

$$H^0(\Delta_X, H^1(I_x, I_x)) \to H^2(\Delta_X, H^0(I_x, I_x)).$$

Proof of Proposition 6 (continued)

Hence we have a commutative diagram

$$\begin{array}{ccc} H^0(\Delta_X, H^1(I_x, I_x)) & \longrightarrow & H^2(\Delta_X, H^0(I_x, I_x)) \\ \downarrow \wr & & \downarrow \wr \\ \mathrm{Hom}(I_x, I_x) & \longrightarrow & \mathrm{Hom}(M_X, I_x). \end{array}$$

In this case, the element $1 \in H^0(\Delta_X, H^1(I_x, I_x))$ determines a natural isomorphism $M_X \xrightarrow{\sim} I_x$. This proves Proposition 6.

Definition 7

Let k be a field of characteristic 0. We say that k is **Kummer-faithful** (resp. **torally Kummer-faithful**) if one of the following equivalent conditions hold for every semi-ableian variety (resp. torus) A defined over an arbitrary finite extension k_H of k corresponding to the open subgroup $H \subset G_k$:

(i) We have

$$\bigcap_{n\geq 1} nA(k_H) = \{0\}.$$

(ii) The Kummer map

$$A(k_H) \to H^1(H, \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, A(\bar{k})))$$

is injective.

Kummer Classes

Proposition 8

Let X be a proper hyperbolic curve over a Kummer-faithful field k. And let $U \subset X$ be an open subcurve. We write κ_U for the Kummer map

$$\kappa_U : \Gamma(U, \mathcal{O}_U^\times) \to H^1(\Pi_U, M_X).$$

Moreover, for each integer d, we write J^d for the moduli space of degree d line bundles on X and $J := J^0 = \mathrm{Jac}_X$. Then the followings hold:
(i) The Kummer map κ_U is injective.
(ii) For each $x \in X(k)$, we write $s_x : G_k \to \Pi_X$ for the section determined by x. We write $t_x : G_k \xrightarrow{s_x} \Pi_X \twoheadrightarrow \Pi_{J^1}$. Then for any divisor D on X such that $\mathrm{Supp}(D) \subset X(k)$, there exists a section $t_D : G_k \to \Pi_{J^d}$. If $d = 0$, then t_D coincide with the section determined by $0 \in J(k)$ if and only if D is principal.

Proposition 8 (continued)

(iii) Write $U := X \setminus S$ for some $S \subset X(k)$ a finite subset. Then restricting cohomology classes of Π_U to I_x for each $x \in S$ yields an exact sequence

$$1 \to (k^\times)^\wedge \to H^1(\Pi_U, M_X) \to \bigoplus_{x \in S} \widehat{\mathbb{Z}}.$$

Moreover, the image of $\Gamma(U, \mathcal{O}_U^\times)$ in $H^1(\Pi_U, M_X)/(k^\times)^\wedge$ via κ_U coincide with the inverse image of the submodule

$$\bigoplus_{x \in S} \mathbb{Z} \subset \bigoplus_{x \in S} \widehat{\mathbb{Z}}$$

determined by the principal divisors with support in S.

Proof of Proposition 8

Since k is Kummer-faithful, hence it is torally Kummer-faithful, which implies that the function field $k(X)$ is also Kummer-faithful, i.e. the natural map

$$k(X)^\times \to H^1(k(X), \widehat{\mathbb{Z}}(1))$$

is injective. On the other hand, we have

$$k(X)^\times = \varinjlim_{U} \Gamma(U, \mathcal{O}_U^\times)$$

and

$$\varinjlim_{U} H^1(\Pi_U, M_X) = H^1(k(X), \widehat{\mathbb{Z}}(1)).$$

Hence the Kummer map κ_U is injective, this proves assertion (i).

Proof of Proposition 8

For assertion (ii), t_D can be constructed by considering the map induced by maps $J^1 \to J^d$ induced by supports of D:

$$\Pi_{J^1} \times_{G_k} \cdots \times_{G_k} \Pi_{J^1} \to \Pi_{J^d}.$$

Now we assume that $d = 0$. The Kummer sequence for J induces the following isomorphism

$$H^1(k, \Delta_J) \xrightarrow{\sim} J(k)^\wedge$$

together with the natural inclusion $J(k) \hookrightarrow J(k)^\wedge$ (since k is Kummer-faithful). Thus, we have D is principal $\iff$ D corresponds to $0 \in J(k)$ $\iff$ t_D coincide with the section determined by 0. This proves assertion (ii).

Kummer Classes

Proof of Proposition 8

Consider the Hoschild-Serre spectral sequence for $\Pi_U \twoheadrightarrow \Pi_X$ and $\Pi_U^{\text{c-cn}} \twoheadrightarrow \Pi_X$, we conclude that the following inflation map is an isomorphism:

$$\text{inf} : H^1(\Pi_U^{\text{c-cn}}, M_X) \xrightarrow{\sim} H^1(\Pi_U, M_X).$$

Then the exactness of the sequence in assertion (iii) follows from the Hoschild-Serre spectral sequence for $\Pi_U^{\text{c-cn}} \twoheadrightarrow \Pi_X$ together with the fact $H^0(k, \Delta_J) = 0$. The second part of assertion (iii) follows from the exact sequence in assertion (iii). This proves Proposition 8.

Kummer Classes

Definition 9

Let k be a field of characteristic 0, we write $\bar{k}_{\mathrm{NF}}$ for the algebraic closure of $\mathbb{Q}$ in $\bar{k}$.
(i) We say that X is an NF-curve if $X_{\bar{k}}$ can be defined over $\bar{k}_{\mathrm{NF}}$.
(ii) If X is an NF-curve. A point in $X(\bar{k})$ is said to be an NF-point if it descents to $\bar{k}_{\mathrm{NF}}$.
Moreover, a rational function on $X_{\bar{k}}$ is said to be an NF-rational function if it descents to $\bar{k}_{\mathrm{NF}}$. An NF-constant is a constant NF-function.

Kummer Classes

Proposition 10

Assume that we are in the situation of Proposition 8. Suppose in addition that U is an NF-curve. We write

$$\mathcal{P}_U \subset H^1(\Pi_U, M_X)$$

for the inverse image of the submodule $\bigoplus_{x \in S} \mathbb{Z} \subset \bigoplus_{x \in S} \widehat{\mathbb{Z}}$. Then the followings hold:
(i) A class $\eta \in \mathcal{P}_U$ is the Kummer class (i.e. lies in $\mathrm{im}(\kappa_U)$) of a non-constant NF-rational function if and only if there exists some positive integer n such that $\eta^\dagger := \eta^n$ and NF-points $x_i \in U(k(x))$ where $i = 1, 2$, such that the cohomology classes

$$\eta^\dagger|_{x_i} := s_{x_i}^*(\eta^\dagger) \in H^1(G_{k(x)}, M_X)$$

satisfy $\eta^\dagger|_{x_1} = 1$ and $\eta^\dagger|_{x_2} \neq 1$.

Proposition 10 (coninued)

(ii) Suppose that there exists some non-constant NF-rational functions in $\Gamma(U, \mathcal{O}_U^\times)$. Then a class $\eta \in \mathcal{P}_U \cap H^1(G_k, M_X)$ is the Kummer class of an NF-constant in $k^\times$ if and only if there exists a non-constant NF-rational function $f \in \Gamma(U, \mathcal{O}_U^\times)$ and an NF-point $x \in U(k(x))$ such that

$$\kappa_U(f)|_x = \eta|_{G_{k(x)}}.$$

Proof of Proposition 10

Notice that U and hence X are NF-curves. Hence $X_{\bar{k}}$ descents to $X_{\bar{k}_{\mathrm{NF}}}$. Since X is proper, then for any non-constant rational function f on $X_{\bar{k}_{\mathrm{NF}}}$, f determines a surjective morphism $\phi_f : X_{\bar{k}_{\mathrm{NF}}} \twoheadrightarrow \mathbb{P}^1_{\bar{k}_{\mathrm{NF}}}$ which induces a surjective map $\tilde{\phi}_f : X_{\bar{k}_{\mathrm{NF}}}(\bar{k}_{\mathrm{NF}}) \to \mathbb{P}^1_{\bar{k}_{\mathrm{NF}}}(\bar{k}_{\mathrm{NF}})$.
Consider assertion (i). Let $\eta \in \mathcal{P}_U$ be the Kummer class of some non-constant NF-rational function f. Then by surjectivity of $\tilde{\phi}_f$, we take $x_1 \in U(k(x))$ be such that $f(x_1) = 1$ and $x_2 \in U(k(x))$ be such that $f(x_2) \notin \mu_\infty$. This verifies the existence of $x_1, x_2 \in U(k(x))$ and n.
Conversely, suppose that $\eta \in \mathcal{P}_U$ admits $\eta^\dagger = \eta^n$ for some positive integer n and NF-points $x_1, x_2 \in U(k(x))$ satisfying $\eta^\dagger|_{x_1} = 1$ and $\eta^\dagger|_{x_2} \neq 1$. Then Proposition 8 (iii) implies that $\eta^\dagger$ comes from a rational function f on U. With $f(x_1) \neq f(x_2)$ hence non-constant. Furthermore, since U is an NF-curve, f descents to $\bar{k}_{\mathrm{NF}}$, hence f is a non-constant NF-rational function. This verifies assertion (i).
Assertion (ii) is an immediate consequence of Proposition 8 (iii)

Reconstruction of Curves

Let X be a hyperbolic orbicurve of strictly Belyi type over a sub-p-adic field k for some prime number p.
Let $1 \to \Delta \to \Pi \to G \to 1$ be an extension of profinite groups isomorphic to

$$1 \to \Delta_X \to \Pi_X \to G_k \to 1.$$

Reconstruction of Curves

Theorem 11

There exists a group-theoretic reconstruction of fields $\bar{k}_{\mathrm{NF}}(\Pi)$ and $\bar{k}(\Pi)$ such that the following diagram commutes

$$\begin{array}{ccc} \bar{k}_{\mathrm{NF}}(\Pi) & \hookrightarrow & \bar{K}_{\mathrm{NF}}(\Pi) \\ \downarrow \wr & & \downarrow \wr \\ \bar{k}_{\mathrm{NF}} & \hookrightarrow & \bar{k}_{\mathrm{NF}}(Z) \end{array}$$

where Z denotes the smooth compactification of a finite étale cover $Y \to X$ (i.e. the cover appear in the definition of strictly Belyi type).

Proof of Theorem 11

Step 1: For some open subgroup $H \subset \Pi$, by using Belyi cuspidalisation developed in the previous talk, one can group-theoretically reconstruct the following surjection

$$H^+ \twoheadrightarrow H$$

where for H corresponding to some finite étale cover $Y \to X$ which is an NF-curve, and H^+ corresponding to some open subcurve $U \subset Y$. In particular, from Π, we can group-theoretically reconstruct H^+ together with decomposition subgroups at NF-points.

Proof of Theorem 11

Step 2: Assume that $g_Y \geq 2$. We write $Z := Y^+$ for the smooth compactification of Y. By Proposition 6, one constructs the following isomorphism

$$I_z \xrightarrow{\sim} \Lambda(H^+) := M_Z$$

where $z \in (Z \setminus U)(k_Z)$, k_Z is the base field for Z.

Proof of Theorem 11

Step 3: By Proposition 6, together with step 2, one reconstructs the group

$$\mathcal{P}(H^+) \xrightarrow{\sim} \mathcal{P}_U \subset H^1(\Pi_U, \Lambda(\Pi_U))$$

determined by the cuspidal principal divisors.

Proof of Theorem 11

Step 4: By using Proposition 10, together with step 1. One reconstruct the groups

$$\bar{k}^{\times}_{\mathrm{NF}}(\Pi) \hookrightarrow \bar{K}^{\times}_{\mathrm{NF}}(\Pi) \hookrightarrow \varinjlim_{H^+} H^1(H^+, \Lambda(H^+))$$

where H^+ ranges over all possible H^+ (c.f. Step 1) arising from removing NF-points of $Z_{k'}$ for some finite extension k'/k_Z.

Proof of Theorem 11

Step 5: By using Proposition 3, together with step 1, we obtain the additive structures on $k_{\mathrm{NF}}^{\times}(\Pi) \cup \{0\}$ and $\bar{K}_{\mathrm{NF}}^{\times}(\Pi) \cup \{0\}$.
This completes Theorem 11.

Reconstruction of Curves

Corollary 12

Let X be a hyperbolic curve over a number field k of strictly Belyi type. Let $\Pi \xrightarrow{\sim} \Pi_X$ be a profinite group. There is a group-theoretic reconstruction of a field $\bar{k}(\Pi) \xrightarrow{\sim} \bar{k}(X)$.

Further Questions

Natural Questions

(i) Can one develop a version of Theorem 11 respect G_k-open homomorphisms for some sub-p-adic field k?

(ii) Assume that X/k_1 and Y/k_2 be hyperbolic curves of strictly Belyi type over number fields k_1, k_2. Let us assume that every open homomorphism $G_{k_1} \to G_{k_2}$ arises from field embedding. Can one establish a version of Corollary 12 that respect open homomorphisms $\Pi_X \to \Pi_Y$?

(iii) Can one develop a version of Theorem 11 for curves over finite fields?

The End