Cuspidalisations in Anabelian Geometry

Lecture 1

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Introduction

First Question

What is anabelian geometry?

Introduction

Grothendieck

Anabelian geometry is the study of schemes from their étale fundamental groups. More precisely, if X/K is a scheme over a finitely generated field K, then the more complicated $\pi_1^{\text{ét}}(X_{\bar{K}})$ is, the more information we can unpack from $\pi_1^{\text{ét}}(X)$. A scheme X is called anabelian if its isomorphy-type is determined uniquely from the isomorphy-type of $\pi_1^{\text{ét}}(X)$.

Introduction

Natural Question

What schemes are anabelian?

Grothendieck Conjectures

Isom-form: Hyperbolic curves over finitely generated fields over $\mathbb{Q}$ are anabelian.
Hom-form: If X, Y are hyperbolic curves, then every open homomorphism $\pi_1^{\text{ét}}(X) \to \pi_1^{\text{ét}}(Y)$ is geometric, i.e. arises uniquely from a dominant morphism $X \to Y$.
Section conjecture: Let X/K be a hyperbolic curve, K be a finitely generated field over $\mathbb{Q}$. Then every section $s : G_K \to \pi_1^{\text{ét}}(X)$ to the natural projection $\pi_1^{\text{ét}}(X) \to G_K$ is geometric, i.e. arises uniquely from a rational point $x : \text{Spec}(K) \to X$.

Known results

Neukirch,Uchida,Ikeda,Iwasawa,Pop,etc

If k is a finitely generated field over its prime field, then $\mathrm{Spec}(k)$ is anabelian.

Tamagawa

Let X, Y be affine hyperbolic curves over finitely generated fields over $\mathbb{Q}$, then the following map

$$\mathrm{Isom}(X, Y) \to \mathrm{Isom}(\pi_1^{\text{ét}}(X), \pi_1^{\text{ét}}(Y))/\mathrm{Inn}(\pi_1^{\text{ét}}(Y))$$

is bijective.

Known results

Mochizuki

Let k be a subfield of a finitely generated field over $\mathbb{Q}_p$ for some prime number p. Let X, Y be hyperbolic curves over k, then the following map

$$\mathrm{Hom}^{\mathrm{dom}}_{\mathrm{Spec}(k)}(X, Y) \to \mathrm{Hom}_{G_K}(\pi_1^{(p)}(X), \pi_1^{(p)}(Y))/\mathrm{Inn}(\Delta_Y)$$

is bijective.

Absolute vs relative

Relative Anabelian Geometry

Results like Mochizuki's theorem, we assume everything is defined over the same base field.

Absolute Anabelian Geometry

Results like Tamagawa's theorem, we do not need to assume everything is defined over the same base field.

Absolute anabelian geometry

Observation by Mochizuki

The absolute version of Hom-form for hyperbolic curves over sub-p-adic fields (more precisely, over p-adic local fields) is wrong! But the absolute version of Isom-form for hyperbolic curves over p-adic local fields is plausible.

Cuspidalisation

Let X be a proper hyperbolic curve over a p-adic local field k. Let S be a finite subset of closed points of X. We write

$$U_S := X \setminus S$$

for the open subscheme of X. Notice that we have an open immersion

$$U_S \hookrightarrow X$$

which determines a surjection

$$\Pi_{U_S} \twoheadrightarrow \Pi_X.$$

Cuspidalisation

The ultimate goal is to reconstruct the profinite group Π_{U_S} from Π_X. But this turns out to be too difficult. But we have

Mochizuki (Cuspidalisation)

We can reconstruct some quotient $\Pi_{U_S} \twoheadrightarrow \Gamma_{U_S}$ from Π_X which surjects onto Π_X.

Mono-anabelian geometry

A typical example of a bi-anabelian result

Let X and Y be hyperbolic curves, then $\Pi_X \xrightarrow{\sim} \Pi_Y \implies X \xrightarrow{\sim} Y$.

A typical example of a mono-anabelian result

Let Π be an abstract profinite group isomorphic to Π_X for some hyperbolic curve X.
Then there is a group-theoretic reconstruction to a field $K(\Pi)$, isomorphic to $K(X)$.

Structure of the study group

Schedule

Week 1: Introduction, review of basic fact from algebraic curves.
Week 2: Brief introduction to stacks.
Week 3: Brief introduction to étale fundamental groups.
Week 4 (Zhongpeng): Some Group-theoretic properties of fundamental groups of hyperbolic orbicurves.
Week 5 (Zhongpeng): Group-theoretic reconstruction of decomposition groups.
Week 6 (Sean): The theory of chains.
Week 7-10: Break
Week 11: Applications of chain operations.
Week 12 (Zhongpeng): Elliptic cuspidalisation.
Week 13: Belyi cuspidalisation.
Week 14: Cyclotomic synchronisation and Kummer classes.
Week 15: Mono-anabelian reconstruction of curves.

Introduction to Algebraic Curves

Definition 1.1

An algebraic curve, or a curve for short, is a 1 dimensional separated integral scheme of finite type over a field k. Essentially, a 1 dimensional algebraic variety.

Definition 1.2

A curve X is said to be proper if the structure morphism

$$X \to \mathrm{Spec}(k)$$

is a proper morphism, i.e. if for any algebraic variety X' over k, the projection map

$$X \times_{\mathrm{Spec}(k)} X' \to X'$$

is a closed map.

Definition 1.3

A curve X is said to be projective if it is proper. Otherwise, X is said to be affine.

Caution

Properness $\iff$ projectivity $\iff$ completeness holds only for curves! Hence Definition 1.3 only make sense for curves.

Generalities

Definition 1.4

Let $x \in X$ be a closed point, then the residue field at x is defined to be

$$\kappa(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x.$$

Fact

$\kappa(x)$ is always a finite extension of the base field k.

Definition 1.5

Let X be a curve, and $x \in X$ be a closed point. We say that X is smooth at x if $\mathcal{O}_{X,x}$ is a regular local ring (a local ring R is said to be regular if $\dim(R) = \dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$). A curve is said to be smooth if every closed point x of X is regular.

Proposition 1.6

Let X be a curve over a perfect field k, and $x \in X$ be a closed point. Then the followings are equivalent:
(i) X is smooth at x.
(ii) $\mathcal{O}_{X,x}$ is normal (i.e. normal means it is integrally closed in its field of fractions).
(iii) $\mathcal{O}_{X,x}$ is a DVR.
(iv) The ideal sheaf $\mathcal{I}_x$ is an invertible $\mathcal{O}_X$-module.

Definition/Proposition 1.7

Let X be a smooth curve over k. There exists a unique smooth projective curve $\overline{X}$ such that $X \hookrightarrow \overline{X}$ as a dense open subscheme, moreover, the image of X in $\overline{X}$ is the complement of finitely many of closed points of $\overline{X}$. In this case, we say that $\overline{X}$ is the smooth compatification of X.

Definition 1.8

Let X be a smooth curve, we define the genus of X as follows:
(i) If X is projective, then

$$g_X := \dim_k H^1(X, \mathcal{O}_X).$$

(ii) If X is affine, then

$$g_X := g_{\overline{X}}.$$

Generlities

Theorem 1.9

The category of smooth projective curves over $\mathbb{C}$ is equivalent to the category of compact Riemann surfaces.

Definition 2.1

Let X be a curve over a field k. We write

$$\mathrm{Div}(X) := \text{the free abelian group generated by closed points of } X.$$

We call this free abelian group the divisor group of X. An element $D \in \mathrm{Div}(X)$ is called a (Weil)divisor, we have

$$D = \sum_{x \in X} n_x [x]$$

where $n_x \in \mathbb{Z}$ and for almost all $x \in X$, $n_x = 0$.

Divisors

Definition 2.2

A divisor D on X is said to be effective or positive if $n_x \geq 0$ for all $x \in X$.

Definition 2.3

We define $\deg(D) := \sum_x n_x$ to be the degree of D. Moreover, we write

$$\mathrm{Div}^0(X) := \{D \in \mathrm{Div}(X) : \deg(D) = 0\}.$$

Divisors

Definition 2.4

If X is a smooth curve over k. Let $f \in k(X)^\times$ be a non-zero rational function on X. We define

$$\mathrm{div}(f) := \sum_x \mathrm{ord}_x(f)[x] \in \mathrm{Div}(X).$$

A divisor D is said to be principal if $D = \mathrm{div}(f)$ for some $f \in k(X)^\times$. And two divisors D, D' are said to be linearly equivalent (written as $D \sim D'$) if $D - D'$ is principal.

Divisors

Definition 2.5

We denote by $\mathrm{Pic}(X)$ for the quotient of $\mathrm{Div}(X)$ by the subgroup generated by principal divisors. We call this group the Picard group of X (or divisor class group).

Fact

One verifies easily that we have the following exact sequence

$$1 \to H^0(X, \mathcal{O}_X^\times) \to k(X)^\times \xrightarrow{\mathrm{div}} \mathrm{Div}(X) \to \mathrm{Pic}(X) \to 1.$$

Divisors

Definition 2.6

Let D be a divisor on X, the sheaf $\mathcal{O}_X(D)$ is defined as

$$\mathcal{O}_X(D)(U) := \{f \in k(X)^\times : \mathrm{div}(f)|_U + D|_U \text{ is effective}\} \cup \{0\}$$

where U ranges over all open subschemes of X.

Definition 2.7

Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module on X. We can associated a divisor to $\mathcal{L}$ as follows: Let $s \in H^0(X, \mathcal{L} \otimes_{\mathcal{O}_X} k(X))$ and we set $v_x(s) := \mathrm{ord}_x(s)$ for all closed point x. We then define $D(s) := \sum_x v_x[x]$.

Proposition 2.8

Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module on X. Then there is a canonical isomorphism of sheaves

$$\mathcal{L} \xrightarrow{\sim} \mathcal{O}_X(D(s))$$

for every section s as in Definition 2.7. In particular, the linear equivalence class of $D(s)$ depends only on the isomorphism class of $\mathcal{L}$.

Definition 2.9

Let X be a smooth curve, then the canonical bundle ω_X is defined to be the sheaf of Kähler differentials $\Omega^1_{X/k}$ on X. Moreover, a canonical divisor K_X of X is a divisor associated to ω_X, this is well-defined up to linear equivalence.

Riemann-Roch Theorem

Theorem 3.1 (Riemann-Roch Theorem)

Let X be a smooth projective curve over a field k of characteristic 0. Let D be a divisor on X. Then the following equality holds:

$$\dim_k(H^0(X, \mathcal{O}_X(D))) + \dim_k(H^0(X, \mathcal{O}_X(K_X - D))) = \deg(D) + g_X - 1.$$

Definition 3.2

Let $f : X \to Y$ be a non-constant morphism of degree d between smooth projective curves over a field k of characteristic 0. Let $x \in X$ be a closed point, and $y := f(x)$ be the image of x. Then f induces an injective morphism (this is not trivial):

$$f^{\#} : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}.$$

We say that f ramifies at x if $\mathfrak{m}_y \mathcal{O}_{X,x} = \mathfrak{m}_x^{e_x}$ for some integer $e_x > 1$, in this case, we call the integer e_x the ramification index. If $e_x = 1$, we say that f is unramified at x.

Application 3.3 (Riemann-Hurwitz formula)

Suppose we are in the situation as in Definition 3.2. The ramification divisor of f on X is defined to be $\mathcal{R} := \sum_x (e_x - 1)[x]$. The Riemann-Hurwitz formula asserts that

$$2g_X - 2 = d(2g_Y - 2) + \deg(\mathcal{R}).$$

Definition 4.1

We define $\mathrm{Pic}^0(X) := \mathrm{Div}^0(X)/k(X)^\times$ for the degree 0 Picard group of X.

Definition 4.2

Let S be any scheme over k. We define $\underline{\mathrm{Pic}}_{X/k}$ for the functor sending S to $\mathrm{Pic}(X_S)$, where $X_S := X \times_{\mathrm{Spec}(k)} S$.

Theorem 4.3

The functor $\underline{\mathrm{Pic}}_{X/k}$ is representable by a scheme locally of finite type over k. In particular, it admits the following decomposition into connected components:

$$\underline{\mathrm{Pic}}_{X/k} = \bigsqcup_{d \in \mathbb{Z}} \mathrm{Pic}^d_{X/k}$$

where $\mathrm{Pic}^d_{X/k}$ is the moduli space of line bundles of degree d on X (which is in fact a scheme!).

Jacobian of curves

Definition 4.4

Let X be a smooth projective curve, then the Jacobian of X is defined to be

$$\mathrm{Jac}_X := \mathrm{Pic}^0_{X/k}.$$

Theorem 4.6

Let X be a smooth projective curve over a field k. Then the followings hold:
(i) The Jacobian Jac_X is an abelian variety over k of dimension g_X.
(ii) For each finite separable extension K/k, we have a natural isomorphsim

$$\mathrm{Jac}_X(K) \xrightarrow{\sim} \mathrm{Pic}^0(X_K).$$

The End