Cusipidalisations in Anabelian Geometry

Week 2: Moduli Stack of Curves

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Introduction

Aim of today

We are going to answer the following three questions:

1. What is a stack?
2. What is a moduli space?
3. What is $\mathcal{M}_{g,r}$?

Definition 1.1

Let $\mathcal{A}$ be a category, the **Grothendieck topology** on $\mathcal{A}$ is defined as follows:
Let U be an object of $\mathcal{A}$, we associated a set T consisting of sets $\mathrm{Cov}(U) := \{\varphi_i : U_i \to U\}_{i \in I}$ (called covers/coverings) where U_i are objects in $\mathcal{A}$ such that the following axioms hold:
(i) For any isomorphism ϕ, one has $\{\phi\} \in T$.
(ii) If $\{U_i \to U\} \in T$ and $\{V_j \to U_i\} \in T$, then $\{V_{i,j} \to U\} \in T$.
(iii) If $\{U_i \to U\} \in T$ and $V \to U$ is a morphism, then $V \times_U U_i$ exists and $\{V \times_U U_i \to U\} \in T$.
A category $\mathcal{A}$ equipped with a Grothendieck topology is called a **site**.

Stacks

Example 1.2

Let X be a topological space, then the category of open subsets of X, denoted by $\mathcal{T}(X)$, forms a site.

Example 1.3

Let X be a scheme, we write $X_{\text{ét}}$ for the category consisting of schemes U étale over X. Morphisms in this category are just morphisms over X, but those X-morphisms are already étale, hence morphisms in this category are étale morphisms over X. We call $X_{\text{ét}}$ the **small étale site on** X. In this case, we define $\text{Cov}(U) := \{U_i \to U : \bigsqcup_i U \twoheadrightarrow U\}$.

Stacks

Example 1.4

The **big étale site** is the category of all scheme Sch, where for any object U, we define the cover associated to U by $\mathrm{Cov}(U) := \{U_i \to U \text{ étale} : \bigsqcup_i U_i \twoheadrightarrow U\}$. We denote by $\mathrm{Sch}_{\text{ét}}$ for this site.

Definition 1.5

Let $\mathcal{S}$ be a site, a **Presheaf of sets on** $\mathcal{S}$ is a contravariant functor

$$\mathcal{F} : \mathcal{S} \to \mathbf{Set}.$$

If moreover, for every object S in $\mathcal{S}$, and coverings $S_i, S_j \to S$, the following sequence is exact:

$$\mathcal{F}(S) \to \prod_i \mathcal{F}(S_i) \rightrightarrows \prod_{i,j} \mathcal{F}(S_i \times_S S_j)$$

we say that $\mathcal{F}$ is a sheaf on $\mathcal{S}$.

Example 1.6

If $X \to S$ is a morphism of schemes, then the functor of points h_X is a sheaf on $S_{\text{ét}}$. Hence schemes are naturally sheaves on sites.

Stacks

Definition 1.7

Let $\mathcal{F}, \mathcal{G}, \mathcal{G}'$ be presheaves on a site $\mathcal{S}$, suppose we have the following diagram

$$\begin{array}{ccc} & & \mathcal{G}' \\ & & \downarrow \beta \\ \mathcal{F} & \xrightarrow{\alpha} & \mathcal{G} \end{array}$$

then the **fibre product of** α and β is the presheaf on $\mathcal{S}$ given by:

$$\mathcal{F} \times_{\mathcal{G}} \mathcal{G}' : \mathcal{S} \to \textbf{Set}$$

$$S \mapsto \mathcal{F}(S) \times_{\mathcal{G}(S)} \mathcal{G}'(S).$$

Definition 1.8

Let $p : \mathcal{X} \to \mathcal{S}$ be a functor. We say the pair $(\mathcal{X}, p)$ (or $\mathcal{X}$ for short if there is no ambiguity) a **prestack** if

(i) For any objects S, T in $\mathcal{S}$, and any morphism $\sigma : S \to T$. If an object B in $\mathcal{X}$ is mapped to T (i.e. $p(B) = T$), then there exists some object A in $\mathcal{X}$ such that $p(A) = S$ and some morphism $\tau : A \to B$ in $\mathcal{X}$ such that $p(\tau) = \sigma$.

(ii) Take any objects R, S, T in $\mathcal{S}$ with morphisms $R \xrightarrow{\sigma'} S \xrightarrow{\sigma} T$. If there are objects A, B, C in $\mathcal{X}$ such that $p(A) = R, p(B) = S, p(C) = D$ and a morphism $\tau : B \to C$ such that $p(\tau) = \sigma$, then there exists a unique morphism $\tau' : A \to B$ such that $p(\tau') = \sigma'$.

Definition 1.9

Let $p : \mathcal{X} \to \mathcal{S}$ be a prestack, and let S be an object in $\mathcal{S}$. Then the **fibre category** $\mathcal{X}(S)$ is defined to be the category whose objects in $\mathcal{X}$ lying above S (i.e. objects X such that $p(X) = S$) with morphisms lying above id_S.

Definition 1.10

A category $\mathcal{G}$ is said to be a **groupoid** if all morphisms in $\mathcal{G}$ are isomoprhisms.

Fact

$\mathcal{X}(S)$ is a groupoid.

Stacks

Example 1.11

Presheaves are prestacks.

Example 1.12

Schemes are prestacks.

Example 1.13

Let X be a scheme, then $\mathrm{QCoh}(\mathrm{X})$ and $\mathrm{Bun}(X)$ are prestacks.

Definition 1.14

Let $\mathcal{S}$ be a site and let $\mathcal{X}$ be a prestack over $\mathcal{S}$. Then $\mathcal{X}$ is said to be a **stack** if the followings hold for all covering $\{S_i \to S\}$ in $\mathcal{S}$:
(i)[*Morphisms glue*] For any objects A, B in $\mathcal{X}$ over S and morphisms $\phi_i : A|_{S_i} \to B$ and $\phi_i|_{S_{ij}} = \phi_j|_{S_{ij}}$, such that there exists a unique morphism $\phi : A \to B$ satisfying $\phi|_{S_i} = \phi_i$.
(ii)[*Objects glue*] For any objects A_i over S_i and isomorphism $\alpha_{ij} : A_i|_{S_{ij}} \xrightarrow{\sim} A_j|_{S_{ij}}$ satisfying the cocycle condition $\alpha_{jk}|_{S_{ijk}} \circ \alpha_{ij}|_{S_{ijk}} = \alpha_{ik}|_{S_{ijk}}$ on S_{ijk}, then there exists an object A over S and isomorphisms $\phi_i : A|_{S_i} \xrightarrow{\sim} A_i$ over id_{S_i} such that $\phi_j|_{S_{ij}} = \alpha_{ij} \circ \phi_i|_{S_{ij}}$ on S_{ij}.

Stacks

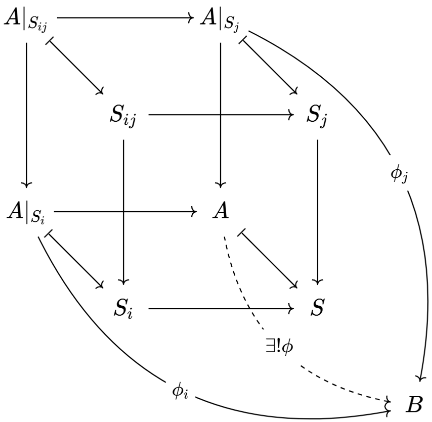

Figure: Morphisms glue

Stacks

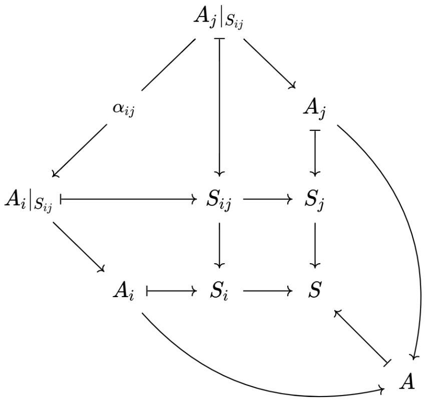

Figure: Objects glue

Stacks

Example 1.15

Sheaves, schemes are stacks.

Example 1.16

Let X be a scheme, $\mathrm{QCoh}(X)$ and $\mathrm{Bun}(X)$ are stacks.

Moduli Stack of Curves

Definition 2.1

Let X be a scheme, a **family of smooth curves** is a smooth proper morphism $f : \mathcal{C} \to X$ such that every geometric fibre of f is a connected curve. Moreover, we say that $f : \mathcal{C} \to X$ is a **family of smooth curves of genus** g if moreover, every geometric fibre of f is a connected curve of genus g.

Definition 2.2

We write $\mathcal{M}_g$ for the category of families of smooth curves of genus g over some scheme X.

Moduli Stack of Curves

Theorem 2.3

Consider the category $\mathcal{M}_g$, the functor

$$p : \mathcal{M}_g \to \mathsf{Sch}_{\text{ét}}$$

$$(\mathcal{C} \to X) \mapsto X$$

is defines a stack over $\mathsf{Sch}_{\text{ét}}$. We call $\mathcal{M}_g$ the **moduli stack of smooth curves of genus** g.

Definition 2.4

Let $\mathcal{X}, \mathcal{Y}$ be (pre)stacks (or sheaves) over $\mathsf{Sch}_{\text{ét}}$. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism. We say that f is **representable by schemes** if for any morphism $T \to \mathcal{Y}$ from a scheme T, the fibre product $\mathcal{X} \times_{\mathcal{Y}} T$ is a scheme.

Moduli Stack of Curves

Definition 2.5

An **algebraic space** is a sheaf $\mathfrak{X}$ on $\mathsf{Sch}_{\text{ét}}$ such that there exists a scheme U and a surjective étale morphism $U \to \mathfrak{X}$ representable by schemes. The morphism $U \to \mathfrak{X}$ is called an **étale presentation**.

Moduli Stack of Curves

Definition 2.6

Let $\mathcal{X}, \mathcal{Y}$ be (pre)stacks over $\mathrm{Sch}_{\acute{e}t}$. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism. We say that f is **representable** if for any morphism $T \to \mathcal{Y}$ from a scheme T, the fibre product $\mathcal{X} \times_{\mathcal{Y}} T$ is an algebraic space.

Definition 2.7

A stack $\mathcal{X}$ over $\mathrm{Sch}_{\acute{e}t}$ is a **Deligne-Mumford stack** if there exists a scheme U and an étale surjective representable morphism $U \to \mathcal{X}$. We call the morphism $U \to \mathcal{X}$ an **étale presentation**.

Definition 2.8

A stack $\mathcal{X}$ over $\mathrm{Sch}_{\acute{e}t}$ is an **algebraic stack** if there exists a scheme U and a smooth surjective representable morphism $U \to \mathcal{X}$. We call a morphism $U \to \mathcal{X}$ a **smooth presentation**.

Moduli Stack of Curves

Theorem 2.9

The moduli stack $\mathcal{M}_g$ of smooth curves of genus g is a Deligne-Mumford stack for $g \geq 2$.

Definition 2.10

Let $\mathcal{X}$ be an algebraic stack, and let $\pi : \mathcal{X} \to X$ be a morphism to an algebraic space X. Then X is called a **coarse moduli space** if
(i) For any algebraic closed field K, there is a bijection

$$\mathcal{X}(K) \xrightarrow{\sim} X(K)$$

from the set of isomorphism classes of objects over $\mathrm{Spec}(K)$.
(ii) X is universal, i.e. if we have another algebraic space X' and a morphism $\pi' : \mathcal{X} \to X'$, π' factors through X.

Pointed Stable Curves

Definition 3.1

Let X be a smooth curve over a field k. We write

$$\mathrm{Aut}(X, p_i) := \{f : X \xrightarrow{\sim} X : f(p_i) = p_i\}$$

for $p_i \in \{p_1, \ldots, p_r\}$ a finite set of k-rational points of X.

Proposition 3.2

Let X be a connected smooth projective curve over an algebraically closed field k, and $\{p_1, \ldots, p_r\}$ be a finite set of k-valued points. Then $\mathrm{Aut}(X, p_i)$ is finite if $2g + r > 2$.

Definition 3.3

Let X be a curve over k. A **split node** on X is a closed point $x \in X(k)$ such that

$$\mathcal{O}_{X,x}^{\wedge} \xrightarrow{\sim} k[[x,y]]/(xy).$$

A closed point $x \in X$ is a **node** if there exists a split node $\bar{x} \in X_{\bar{k}}$ over x. We say that X is a **nodal curve** if every closed point $x \in X$ is either smooth or nodal.

Pointed Stable Curves

Proposition 3.4

Let X be a connected nodal projective curve over an algebraically closed field, then

$$g_X = g_{\widetilde{X}} + \text{number of nodes}$$

where $\widetilde{X} \to X$ is the normalisation.

Pointed Stable Curves

Definition 3.5

Let X be a curve over an algebraically closed field with an ordered collection of closed points $\{p_1, \ldots, p_r\} \subset X$ is called an r-**pointed curve**, and each point in $\{p_1, \ldots, p_r\}$ is called a **marked point**. A closed point $x \in X$ is called a **special point** if either x is marked or x is nodal.

Pointed Stable Curves

Definition 3.6

Let $(X, \{p_1, \ldots, p_r\})$ be an r-pointed connected projective curve of genus g over an algebraically closed field k. We say that X is **stable** if
(i) $p_1, \ldots, p_r \in X(k)$ are smooth.
(ii) X is not of genus 1 without marked points, i.e. $(g, r) \neq (1, 0)$.
(iii) Every smooth irreducible rational subcurve $\mathbb{P}^1 \subset X$ contains at least 3 special points.
If we replace 3 in (iii) by 2, then we call X **semi-stable**. If only (i),(ii) hold, we call X **pre-stable**.

Pointed Stable Curves

Definition 3.7

A **family of r-pointed (stable)curves** over a scheme S is a proper, flat and finitely presented morphism $\mathcal{C} \to S$ of algebraic spaces with r sections $\sigma_1, \ldots, \sigma_r : S \to \mathcal{C}$ such that every geometric fibre is a (stable)curve.

Pointed Stable Curves

Definition 3.8

The **moduli stack of all** r**-pointed curves of genus** g $\mathcal{M}^{\text{all}}_{g,r}$ is defined as follows: an object of $\mathcal{M}^{\text{all}}_{g,r}$ is a family of curves of genus g over a scheme S with r sections $\sigma_1, \ldots, \sigma_r : S \to C$. A morphism $(C \to S, \{\sigma_i\}) \to (C' \to S', \{\sigma'_i\})$ is a cartesian diagram

$$\begin{array}{ccc} C & \longrightarrow & C' \\ \sigma_i \uparrow\downarrow & & \sigma'_i \uparrow\downarrow \\ S & \longrightarrow & S'. \end{array}$$

Pointed Stable Curves

Theorem 3.9

$\mathcal{M}^{\mathrm{all}}_{g,r}$ is an algebraic stack locally of finite type over $\mathrm{Spec}(\mathbb{Z})$.

Corollary 3.10

We write $\overline{\mathcal{M}}_{g,r}$ for the full subcategory of $\mathcal{M}^{\mathrm{all}}_{g,r}$ consisting of r-pointed family of stable curves. Moreover we write $\mathcal{M}_{g,r}$ for the full subcategory of $\overline{\mathcal{M}}_{g,r}$ consisting of r-pointed family of smooth curves. Then $\mathcal{M}_{g,r} \subset \overline{\mathcal{M}}_{g,r} \subset \mathcal{M}^{\mathrm{all}}_{g,r}$ are open immersions. In particular, $\mathcal{M}_{g,r}$ and $\overline{\mathcal{M}}_{g,r}$ are algebraic stacks locally of finite type over $\mathrm{Spec}(\mathbb{Z})$.

Pointed Stable Curves

Theorem 3.11

Assuming that $\overline{\mathcal{M}}_{g,r}$ satisfying that $2g + r > 2$, then $\overline{\mathcal{M}}_{g,r}$ is a Deligne-Mumford stack smooth proper irreducible $\mathrm{Spec}(\mathbb{Z})$ of dimension $3g - 3 + r$. Moreover, its coarse moduli space $\overline{M}_{g,r}$ is a projective scheme over $\mathrm{Spec}(\mathbb{Z})$.

The End