Cuspidalisations in Anabelian Geometry

Week 3: Fundamental Groups of Curves

Yu Mao

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Étale Fundamental Groups

Definition 1.1

Let X be a scheme, we define

$$FEt(X)$$

to be the category consisting of objects which are schemes U with a finite étale morphism $U \to X$.

Étale Fundamental Groups

Definition 1.2

Let C be a category, $\mathcal{F} : C \to \mathbf{Set}$ be a functor. Then $(C, \mathcal{F})$ is called a **Galois category** if:

(i) C has finite limits and colimits.

(ii) Every object of C is a finite (possibly empty) coproduct of connected objects. (An object $X \in \mathrm{ob}(C)$ is called connected if it is not initial and for any monomorphism $Y \to X$, either Y is initial or this monomorphism is an isomorphism)

(iii) $\mathcal{F}(X)$ is finite for all objects $X \in C$.

(iv) If f is a morphism in C, then $\mathcal{F}(f)$ is an isomorphism implies that f is an isomorphism.

(v) $\mathcal{F}$ is exact.

In this case, we call $\mathcal{F}$ the **fibre functor** of C.

Étale Fundamental Groups

Theorem 1.3

Let $(C, \mathcal{F})$ be a Galois category, $G := \mathrm{Aut}(\mathcal{F})$. Then

$$\mathcal{F} : C \to \mathbf{FinSet}_G$$

is an equivalence of categories. In particular, we call G the fundamental group of C.

Étale Fundamental Groups

Proposition 1.4

The category $\mathsf{FEt}(X)$ is a Galois category. More precisely, fix a geometric point $\bar{x} : \mathrm{Spec}(\Omega) \to X$, and a finite étale cover $U \to X$, we define $U_{\bar{x}} := U \times_X \mathrm{Spec}(\Omega)$. Then the functor

$$\mathcal{F}_{\bar{x}} : \mathsf{FEt}(X) \to \mathbf{Set}$$

$$U \mapsto |U_{\bar{x}}|$$

where $|U_{\bar{x}}|$ is the underlying topological space of $U_{\bar{x}}$. In this case, we write $\pi_1^{\text{ét}}(X, \bar{x}) := \mathrm{Aut}(\mathcal{F}_{\bar{x}})$ for the fundamental group of $\mathsf{FEt}(X)$. We call $\pi_1^{\text{ét}}(X, \bar{x})$ the **étale fundamental group** of X. We call $\bar{x}$ the **base point** of $\pi_1^{\text{ét}}(X, \bar{x})$.

Étale Fundamental Groups

Corollary 1.5

Let X be a connected scheme and $\bar{x} : \mathrm{Spec}(\Omega) \to X$ be a geometric point.

(i) There is an equivalence of categories:

$$\mathcal{F}_{\bar{x}} : \mathsf{FEt}(X) \to \mathbf{FinSet}_{\pi_1^{\text{ét}}(X, \bar{x})}.$$

(ii) Let $\bar{y} : \mathrm{Spec}(\Omega') \to X$ be another geometric point. Then there is an isomorphism $t : F_{\bar{x}} \xrightarrow{\sim} F_{\bar{y}}$. This isomorphism determines an isomorphism $\pi_1^{\text{ét}}(X, \bar{x}) \xrightarrow{\sim} \pi_1^{\text{ét}}(X, \bar{y})$, this isomorphism is independent from t upto conjugation. In this case, we shall write Π_X for the étale fundamental group of X.

(iii) The assignment $X \mapsto \Pi_X$ is functorial.

Definition 1.6

Let $(C, \mathcal{F})$ be a Galois categroy. An object X in C is called a **Galois object** if $\mathcal{F}(X)$ is an $\text{Aut}(\mathcal{F})$-torsor, i.e. $\text{Aut}(\mathcal{F})$ acts simply transitively on $\mathcal{F}(X)$.

Definition 1.7

Let X be a connected scheme. Let $Y \to X$ be a finite étale cover. We say that $Y \to X$ is a **Galois cover** if Y is a Galois object in $\text{FEt}(X)$. In this case, we write $\text{Gal}(Y/X) := \text{Aut}_X(Y)$ for the Galois group of Y over X.

Étale Fundamental Groups

Lemma 1.8

Let $Y \to X$ be a finite étale cover. Then there exists some finite Galois cover $Y' \to X$ dominantes Y. In particular, we have the Galois correspondence:

$$\{Y \to X \text{ finite étale}\} \leftrightarrow \{H \subset \Pi_X \text{ open}\}$$

and

$$\{Y \to X \text{ finite Galois}\} \leftrightarrow \{H \lhd \Pi_X \text{ open normal}\}.$$

Étale Fundamental Groups

Theorem 1.9

Let X be a connected scheme, then there is a natural isomorphism

$$\Pi_X \xrightarrow{\sim} \varprojlim_{Y \to X \text{ fin.Gal}} \text{Gal}(Y/X).$$

Étale Fundamental Groups

Corollary 1.10

Let X be a geometrically connected variety over a field k. We write
$X_{\bar{k}} := X \times_{\text{Spec}(k)} \text{Spec}(\bar{k})$. There exists an exact sequence

$$1 \to \Pi_{X_{\bar{k}}} \to \Pi_X \to G_k \to 1$$

we call this sequence the **homotopy exact sequence**. In this case, we write $\Delta_X := \Pi_{X_{\bar{k}}}$, we call this group the **geometric fundamental group** of X.

Examples of Étale Fundamental Groups

Theorem 2.1 (Riemann Existence Theorem)

Let X be a geometrically connected variety over a field k of characteristic 0. There exists a natural isomorphism

$$\Delta_X \xrightarrow{\sim} \pi_1^{\text{top}}(X(\mathbb{C}))^{\wedge}.$$

Corollary 2.2

Let X be a projective normal curve over a field k of characteristic 0. And let $U \subset X$ be an open subscheme. Then

$$\Delta_U \xrightarrow{\sim} \langle a_i, b_i, \gamma_j \mid \prod_i [a_i, b_i] \prod_j \gamma_j = 1 \rangle^{\wedge}_{1 \leq i \leq g; 1 \leq j \leq r}$$

where $g := g_X$ and $r := |X \setminus U|$.

Examples of Étale Fundamental Groups

Example 2.3

We assume that everything is defined over k a field of characteristic 0. Then
(i) $\Delta_{\mathbb{A}^n} = 0$ for all $n \geq 1$.
(ii) $\Delta_{\mathbb{P}^n} = 0$ for all $n \geq 1$.

Example 2.4

(i) $\Pi_{\operatorname{Spec}(\mathbb{Z})} = 1$.
(ii) Let F be a number field, then $\Pi_{\operatorname{Spec}(\mathcal{O}_F)}^{\operatorname{ab}} \xrightarrow{\sim} \operatorname{CL}^{\operatorname{fin}}(F)$. Here $\operatorname{CL}^{\operatorname{fin}}(F)$ denotes the narrow class group of F, which is a variant of the ideal class group with infinite primes ignored.
(iii) Let k be a finite extension of $\mathbb{Q}_p$ for some prime number p. Then
$\Pi_{\operatorname{Spec}(\mathcal{O}_k)} \xrightarrow{\sim} G_k^{\operatorname{unr}} \xrightarrow{\sim} \widehat{\mathbb{Z}}$.

Examples of Étale Fundamental Groups

Example 2.5

We assume that everything is defined over k a field of characteristic 0.
(i) Let $X := \mathbb{P}^1 \setminus \{0, 1\}$, then by Corollary 2.2 $\Delta_X \xrightarrow{\sim} \widehat{\mathbb{Z}}$.
(ii) Let $X := \mathbb{P}^1 \setminus \{0, 1, \infty\}$, then by Corollary 2.2, $\Delta_X \xrightarrow{\sim} \widehat{F_2}$.

Belyi's Theorem

Let C be a smooth projective curve over $\mathbb{C}$ and $X := \mathbb{P}^1 \setminus \{0, 1, \infty\}$. Then the followings are equivalent:
(i) C is defined over some $\overline{\mathbb{Q}}$.
(ii) There exists a non-constant morphism $\beta : C \to \mathbb{P}^1$ ramifies at 3 points or less.
(iii) There exists a finite étale morphism $C \to X$.

Examples of Étale Fundamental Groups

Example 2.6

Let $\mathbb{A}^1$ be defined over $\mathbb{F}_p$ for some prime number p. Then $\Delta_{\mathbb{A}^1}$ is non-trivial. Consider the morphism

$$F : \mathbb{A}^1 \to \mathbb{A}^1$$

where F is defined to be the inverse of $\mathrm{Frob}_p - \mathrm{id}$. This gives a non-trivial finite étale cover with Galois group $\mathbb{Z}/p\mathbb{Z}$

Examples of Étale Fundamental Groups

Theorem 2.8

Let A be an abelian variety over a number field k. Then it holds that

$$\Delta_A \xrightarrow{\sim} T(A) := \prod_p T_p(A)$$

where $T_p(A) := \varprojlim_n A[p^n]$ denotes the p-adic Tate module. In particular, every finite étale cover of A is abelian.

Examples of Étale Fundamental Groups

Corollary 2.9

Let X be a smooth projective curve. Then there exists a canonical isomorphism

$$\Delta_X^{\mathrm{ab}} \xrightarrow{\sim} \Delta_J$$

where $J := \mathrm{Jac}_X$ is the Jacobian of X.

Corollary 2.10

The section conjecture does not hold for abelian varieties.

Decomposition Groups

Definition 3.1

Let X be a curve and $Y \to X$ be a finite étale cover. Let $y \in Y$ be a closed point. The **decomposition group at y** is defined as:

$$D_y(Y/X) := \{\sigma \in \text{Aut}_X(Y) : \sigma(y) = y\}.$$

If we take the inverse limit on Y, we obtain

$$D_{\tilde{x}} := \varprojlim_{Y/X} D_y(Y/X)$$

where $\tilde{x}$ is defined to be the point of $\widetilde{X} := \varprojlim_{Y/X} Y$ lying above $x \in X$. We may also write D_x for the decomposition group $D_{\tilde{x}}$, but this is only well-defined up to conjugation.

Definition 3.2

The **inertia subgroup of** $D_{\tilde{x}}$ is defined to be the intersection

$$I_{\tilde{x}} := \Delta_X \cap D_{\tilde{x}}.$$

Proposition 3.3

There is a short exact sequence

$$1 \to I_{\tilde{x}} \to D_{\tilde{x}} \to G_\kappa \to 1$$

where x is the closed point of X corresponds to $\tilde{x}$ and κ is the residue field of x.

Decomposition Groups

Theorem 3.4

Let X be a smooth affine curve, and $\overline{X}$ be its smooth compactification and $x \in \overline{X} \setminus X$ be a cusp. Then
$$I_{\tilde{x}} \xrightarrow{\sim} \widehat{\mathbb{Z}}(1).$$

We call $I_{\tilde{x}}$ a **cuspidal inertia group**.

Tame Fundamental Groups

Definition 4.1

Let X be a connected normal curve over a field k and $U \subset X$ be an open subscheme. Let $Y \to U$ be a finite connected étale cover. We say that Y is **tamely ramified along** $X \setminus U$ if for each $x \in X \setminus U$, the closed point in the normalisation of $\mathrm{Spec}(\mathcal{O}_{X,x})$ in $k(Y)$ has ramification index prime to $\mathrm{char}(\kappa(x))$. We say that Y is **tamely ramified** if for all proper normal curve C containing U as a dense open subscheme, Y is tamely ramified along $C \setminus U$.

Tame Fundamental Groups

Definition 4.2

Write $\mathrm{FEt}(U)$ for the category of finite étale covers of U, we write

$$\mathrm{Ftm}(U)$$

for the full subcategory of $\mathrm{FEt}(U)$ consisting of tamely ramified covers of U. Fix a geometric point $\bar{u}$ of U, by considering the restriction of $\mathcal{F}_{\bar{u}}$ to $\mathrm{Ftm}(U)$, we define the **tame fundamental group of** U as

$$\pi_1^{\mathrm{tame}}(U, \bar{u}) := \mathrm{Aut}(\mathcal{F}_{\bar{u}}|_{\mathrm{Ftm}(U)}).$$

In particular, we shall write Π_U^{tame} for this group.

Tame Fundamnetal Groups

Proposition 4.3

Let X be a geometrically connected normal curve over a field k, $p := \mathrm{char}(k)$. Then

(i) $\Pi_X = \Pi_X^{\mathrm{tame}}$ if $p = 0$.

(ii) $\Pi_X^{\mathrm{tame}} \xrightarrow{\sim} \Pi_X / \overline{\langle I_{\tilde{x}}^{\mathrm{wild}} \rangle}_{x \in X}$ where $I_{\tilde{x}}^{\mathrm{wild}} :=$ pro-p Sylow of $I_{\tilde{x}}$ if $p > 0$.

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