Cuspidalisations in Anabelian Geometry

Week 4: Group-theoretical Characterisations and Hyperbolic Orbicurves

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I. Properties of Profinite Groups

Properties of Profinite Groups

Let G be a Hausdorff topological group, and let $H \subseteq G$ be a closed subgroup. Write

$$Z_G(H) \overset{\text{def}}{=} \{g \in G : g \cdot h = h \cdot g, \forall h \in H\} \subseteq N_G(H) \overset{\text{def}}{=} \{g \in G : g \cdot H \cdot g^{-1} = H\}$$
$$\subseteq C_G(H) \overset{\text{def}}{=} \{g \in G : g \cdot H \cdot g^{-1} \cap H \text{has finite index in} H\}$$

for the centraliser, normaliser and commensurator of H in G, respectively. We shall say G is center-free if $Z_G(G) = \{1\}$.

We shall say H is normally terminal (resp. commensurably terminal) in G, if $H = N_G(H)$ (resp. $H = C_G(H)$). Then

- N is a normal subgroup of $N_G(H)$.
- $Z_G(H)$ and $N_G(H)$ are closed in G.
- H is commensurably terminal $\Rightarrow$ H is normally terminal.

Properties of Profinite Groups

Let $\Sigma \subseteq \mathfrak{Primes}$ be a set of prime numbers, G be a profinite group.

If G admits an open subgroup which is pro-Σ, then we shall say that G is almost pro-Σ. We shall refer to a quotient $G \twoheadrightarrow Q$ as almost pro-Σ-maximal if for some normal open subgroup $N \subseteq G$ with maximal pro-Σ quotient $N \twoheadrightarrow P$, we have $\mathrm{Ker}(G \twoheadrightarrow Q) = \mathrm{Ker}(N \twoheadrightarrow P)$.

If $\Sigma \stackrel{\mathrm{def}}{=} \mathfrak{Primes} \setminus p$ for some $p \in \mathfrak{Primes}$, then we shall write "pro-$(\neq p)$" for "pro-Σ". Write $\widehat{\mathbb{Z}}^{(\neq p)}$ for the maximal pro-$(\neq p)$ quotient of Z. We shall say that G is pro-omissive (respectively, almost pro-omissive) if it is pro-$(\neq p)$ for some $p \in \mathfrak{Primes}$ (respectively, if it admits a pro-omissive open subgroup).

Properties of Profinite Groups

Definition 1.1

(i) Let G be a profinite group. We shall say G is **slim** if $Z_G(H) = \{1\}$ for any open subgroup $H \subseteq G$.

(ii) Let $f : G_1 \to G_2$ be a continuous homomorhism of profinite groups. We shall say $f : G_1 \to G_2$ is **relatively slim** if $Z_{G_2}(Im(H \to G_2)) = \{1\}$ for any open subgroup $H \subseteq G_1$.

Properties of Profinite Groups

Proposition 1.2

Let G be a profinite group, and let H be a closed subgroup of G. Then:

- G is slim $\Leftrightarrow$ any open subgroup of G is center-free.
- $H \subseteq G$ is relatively slim $\Rightarrow$ H and G are slim.
- H is commensurably terminal in G + H is slim $\Rightarrow$ $H \subseteq G$ is relatively slim.
- G is slim $\Rightarrow$ every finite normal subgroup of any open subgroup of G is trivial.

Properties of Profinite Groups

Definition 1.3

We shall say a profinite group G is **topologically finitely generated** if G is topologically generated by a finite set of elements in G.

Proposition 1.4

Let G be a profinite group, and let H be a open subgroup of G. Then:

- G is topologically finitely generated $\Leftrightarrow$ G is a quotient group of $\widehat{F_n}$ for some $n \geq 1$.
- G is topologically finitely generated $\Leftrightarrow$ H is topologically finitely generated.

Properties of Profinite Groups

Proposition 1.5

Let G be a topologically finitely generated profinite group. Then every subgroup of G contains a characteristic open subgroup of G.

Moreover, G admits a basis consisting of characteristic open subgroups, which thus induces a profinite topology on the groups $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$.

Corollary 1.6

Let

$$1 \to G \to \Pi \to J \to 1 \qquad (*)$$

be an exact sequence of profinite groups. Suppose that G is topologically finitely generated and slim. Then $(*)$ determines an outer homomorphism $\rho : J \to \mathrm{Out}(G)$, and we have

$$\Pi \xrightarrow{\sim} \mathrm{Aut}(G) \times_{\mathrm{Out}(G),\rho} J.$$

Properties of Profinite Groups

Definition 1.7

Let G be a profinite group. We shall say G is **elastic** if it holds that every nontrivial topologically finitely generated closed normal subgroup $N \subseteq H$ of an open subgroup $H \subseteq G$ is of finite index in G.

If G is elastic, but not topologically finitely generated, then we shall say that G is **very elastic**.

Example 1.8

If k is a finite field (FF), then $G_k \cong \widehat{\mathbb{Z}}$ is neither slim nor elastic.

Properties of Profinite Groups

Proposition 1.9

Let G be a profinite group, and let H be a open subgroup of G. Then:

- G is elastic $\Rightarrow$ H is elastic.
- H is elastic + G is slim $\Rightarrow$ G is elastic.
- G is very elastic $\Leftrightarrow$ G is nontrivial + every topologically finitely generated closed normal subgroup of H is trivial.

II. Hyperbolic Orbicurves

Hyperbolic Orbicurves

Definition 2.1

A family of **hyperbolic curves** of type (g, r)

$$X \to S$$

is defined to be a morphism which factors $X \hookrightarrow Y \to S$ as the composite of an open immersion $X \to Y$ onto the complement $Y \setminus D$ of a relative divisor $D \subseteq Y$ which is finite étale over S of relative degree r, and a family $Y \to S$ of curves of genus g.

If S is normal, then the pair (Y, D) is unique up to canonical isomorphism. We shall refer to Y (respectively, D) as the **compactification** (respectively, **divisor of cusps**) of X. A family of hyperbolic curves $X \to S$ is defined to be a morphism $X \to S$ such that the restriction of this morphism to each connected component of S is a family of hyperbolic curves of type (g, r) for some integers g, r as above. A family of hyperbolic curve of type $(0, 3)$ will be referred to as a **tripod**.

Hyperbolic Orbicurves

Definition 2.2

If X is a generically scheme-like algebraic stack over a field k that admits a finite étale Galois covering $Y \to X$, where Y is a hyperbolic curve over a finite extension of k, then we shall refer to X as a **hyperbolic orbicurve** over k.

Let $X \hookrightarrow \overline{X}$ be an open immersion of X into a proper hyperbolic orbicurve $\overline{X}$, then we shall refer to $D \stackrel{\text{def}}{=} \overline{X} \setminus X$ as the divisor of cusps of X.

If $X \to Y$ is a dominant morphism of hyperbolic orbicurves, then we shall refer to $X \to Y$ as a partial coarsification morphism if the morphism induced by $X \to Y$ on associated coarse spaces is an isomorphism.

Hyperbolic Orbicurves

Definition 2.3

Let X be a hyperbolic orbicurve over an algebraically closed field; denote its étale fundamental group by Δ_X. We shall refer to the **order** of the [manifestly finite!] decomposition group in Δ_X of a closed point x of X as the order of x. We shall refer to the least common multiple of the orders of the closed points of X as the order of X. Thus X is a hyperbolic curve if and only if the order of X is equal to 1.

Hyperbolic Orbicurves

Example (Semi-elliptic orbicurve)

Let k be a field of characteristic zero, E be an elliptic curve over k with the origin O, we shall refer to $X \stackrel{\text{def}}{=} E \setminus O$ as a **once-punctured elliptic curve**, and refer to the hyperbolic orbicurve obtained as the quotient $X//\{\pm 1\}$ in the sense of stacks as a **semi-elliptic orbicurve** [or an hyperbolic orbicurve of type $(1,1)_{\pm}$] over k.

Hyperbolic Orbicurves

Definition 2.4

We consider hyperbolic curves over an algebraically closed field k of characteristic zero. We call two hyperbolic orbicurves X, Y [over k] **isogenous** if there exists a hyperbolic orbicurve C, together with finite étale morphisms $C \to X$, $C \to Y$.

Hyperbolic Orbicurves

Theorem (Mochizuki)

Let k be an algebraically closed field of characteristic zero. Let X be a hyperbolic curve over k. Let (g, r) be a pair of nonnegative integers satisfying $2g - 2 + r > 0$.
Then (up to isomorphism) there are only finitely many hyperbolic curves over k of type (g, r) that are isogenous to X.
Moreover, if K is an algebraically closed field extension of k, then any curve which is isogenous to X over K is defined over k and already isogenous to X over k.

III. Objects of GFG-type and of GSAFG-type

Objects of GFG-type and of GSAFG-type

Suppose that:

- k is a perfect field, $\overline{k}$ is an algebraic closure of k, $\tilde{k} \subseteq \overline{k}$ a solvably closed Galois extension of k, and $G_k \stackrel{\text{def}}{=} \mathrm{Gal}(\tilde{k}/k)$.
- $X \to \mathrm{Spec}(k)$ is a geometrically connected, smooth, separated algebraic stack of finite type over k.
- $Y \to X$ is a connected finite étale Galois covering which is a [necessarily separated, smooth, and of finite type over k] k-scheme such that $\mathrm{Gal}(Y/X)$ acts freely on some nonempty open subscheme of Y [so X is generically scheme-like].
- $Y \to \overline{Y}$ is an open immersion into a connected proper k-scheme Y such that $\overline{Y}$ is the underlying scheme of a log scheme $\overline{Y}^{\log}$ that is log smooth over k [where we regard $\mathrm{Spec}(k)$ as equipped with the trivial log structure], and the image of Y in $\overline{Y}$ coincides with the interior of the log scheme $\overline{Y}^{\log}$.

Objects of GFG-type and of GSAFG-type

Example 3.1

(i) If $\overline{Y}$ is smooth, D is a union of normal crossings divisors, then we can take $Y = \overline{Y} \setminus D$ [for some suitable choice of log-structure on $\overline{Y}$].
(ii) We can take X to be a hyperbolic orbicurve with finite étale Galois covering Y, where Y is a hyperbolic curve with smooth compatification $\overline{Y}$.

Observatoin (Mochizuki)

(i) Denote by $\mathcal{B}^{\mathrm{tame}}(X, Y)$ the category of finite étale coverings of X whose pull-backs to Y are tamely ramified over [the height one primes of] Y, then $\mathcal{B}^{\mathrm{tame}}(X, Y)$ is a Galois category.
(ii) Denote by $\pi_1^{\mathrm{tame}}(X, Y)$ [or simply $\pi_1^{\mathrm{tame}}(X)$) if $Y \to X$ is fixed] the profinite group associated to the Galois category $\mathcal{B}^{\mathrm{tame}}(X, Y)$.

Objects of GFG-type and of GSAFG-type

Moreover:

- Let $\Sigma \subseteq \mathfrak{Primes}$ be a set of prime numbers, $\pi_1^{\mathrm{tame}}(X_{\bar{k}}) \twoheadrightarrow \Delta_X$ be an almost pro-Σ-maximal quotient of $\pi_1^{\mathrm{tame}}(X_{\bar{k}})$ whose kernel is normal in $\pi_1^{\mathrm{tame}}(X)$, hence determines a quotient $\pi_1^{\mathrm{tame}}(X) \twoheadrightarrow \Pi_X$.
- Suppose that $\pi_1^{\mathrm{tame}}(X) \twoheadrightarrow \mathrm{Gal}(Y/X)$ factors through $\pi_1^{\mathrm{tame}}(X) \twoheadrightarrow \Pi_X$ [$\twoheadrightarrow \mathrm{Gal}(Y/X)$], and that the kernel of $\Delta_X \hookrightarrow \Pi_X \twoheadrightarrow \mathrm{Gal}(Y/X)$ is pro-Σ.
- Hence we obtain natural exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1^{\mathrm{tame}}(X_{\bar{k}}) & \longrightarrow & \pi_1^{\mathrm{tame}}(X) & \longrightarrow & \mathrm{Gal}(\bar{k}/k) & \longrightarrow & 1 \\ & & \downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow & & \\ 1 & \longrightarrow & \Delta_X & \longrightarrow & \Pi_X & \longrightarrow & \mathrm{Gal}(\bar{k}/k) & \longrightarrow & 1. \end{array}$$

Objects of GFG-type and of GSAFG-type

Definition 3.2

(i) We shall refer to any profinite group Δ which is isomorphic to Δ_X [constructed in the above discussion for some choice of data $(k, X, Y \to \overline{Y}, \Sigma)$] as a profinite group of [almost pro-Σ] **GFG-type** [i.e. of geometric fundamental group type].

(ii) We shall refer to any extension of profinite groups $1 \to \Delta \to \Pi \to G \to 1$ which is isomorphic to $1 \to \Delta_X \to \Pi_X \to \mathrm{Gal}(\overline{k}/k) \to 1$ as an extension of profinite groups of [geometrically almost pro-Σ] **AFG-type** [i.e. of arithmetic fundamental group type].

(iii) In the situation of (i) or (ii), we shall refer to any surjection $\pi_1^{\mathrm{tame}}(X_{\overline{k}}) \twoheadrightarrow \Delta_X \xrightarrow{\sim} \Delta$ (resp. any surjection $\pi_1^{\mathrm{tame}}(X) \twoheadrightarrow \Pi_X \xrightarrow{\sim} \Pi$; any isomorphism $\mathrm{Gal}(\overline{k}/k) \xrightarrow{\sim} G$) as a **scheme-theoretic envelope** for Δ (resp. Π; G);

Objects of GFG-type and of GSAFG-type

Let $1 \to \Delta^* \to \Pi^* \to G^* \to 1$ be a extension of AFG-type; let N be the inverse image of Ker $\big(\mathrm{Gal}(\overline{k}/k) \twoheadrightarrow G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\tilde{k}/k)\big)$ via some envelop $\mathrm{Gal}(\overline{k}/k) \xrightarrow{\sim} G^*$.

Suppose that Δ^* is slim and the induced outer action of N on Δ^*_X is trivial. Then for any $n \in N$, choose an arbitrary lift $\tilde{n} \in \Pi^*$ of n, then there exists a unique [since Δ^* is slim] $a \in \Delta^*$ such that $\tilde{n} \cdot a^{-1}$ [which is uniquely determined by n] commutes with Δ^*.

Hence N lifts to a unique closed normal subgroup N_{Π^*} of Π^*, and we obtain an exact sequence $1 \to \Delta^* \to \Pi^*/N_{\Pi^*} \to G^*/N \to 1$.

Objects of GFG-type and of GSAFG-type

Definition 3.3

(i) We shall refer to any extension of profinite groups $1 \to \Delta \to \Pi \to G \to 1$ which is isomorphic to $1 \to \Delta^* \to \Pi^*/N_{\Pi^*} \to G^*/N \to 1$ just constructed as an extension of [geometrically almost pro-Σ] **GSAFG-type** [i.e. of geometrically slim arithmetic fundamental group type].

(ii) We shall refer to any surjection $\pi_1^{\mathrm{tame}}(X_{\overline{k}}) \twoheadrightarrow \Delta$ (resp. $\pi_1^{\mathrm{tame}}(X) \twoheadrightarrow \Pi$; $\mathrm{Gal}(\overline{k}/k) \xrightarrow{\sim} G$) [obtained by composing the related morphism obviously] as a **scheme-theoretic envelope** for Δ (resp. Π; G).

Objects of GFG-type and of GSAFG-type

Remark 3.4

(i) We shall refer to any data $(k, X, Y \to \overline{Y}, \Sigma)$ or $(k, \tilde{k}, X, Y \to \overline{Y}, \Sigma)$ as a collection of **construction data** for the related math object.

(ii) Given construction data, we shall refer to "k" as the **construction data field**, to "X" as the **construction data base-stack** [or **base-scheme**, if X is a scheme], to "Y" as the **construction data covering**, to "$\overline{Y}$" as the **construction data covering compactification**, and to "Σ" as the **construction data prime set**. We shall refer to a portion of some construction data as partial construction data.

(iii) If every prime dividing the order of a finite quotient group of Δ is invertible in k [e.g., if k is of characteristic 0], then we shall refer to the construction data in question as **base-prime**.

IV. Slimness and Elasticity

Slimness and Elasticity

Lemma 4.1

Let k be a p-adic local field with an algebraic closure $\overline{k}$; $G_k \overset{\text{def}}{=} \mathrm{Gal}(\overline{k}/k)$ be the absolute Galois group of k, with maximal pro-p quotient $G_k^{(p)}$; μ_p be the p-th roots of unity in $\overline{k}$.

(i) If $\mu_p \subseteq k$, then for any finite $G_k^{(p)}$-module M annihilated by p and any integer $j \geq 0$, the surjection $G_k \twoheadrightarrow G_k^{(p)}$ induces an isomorphism $H^j(G_k^{(p)}, M) \xrightarrow{\sim} H^j(G_k), M)$.

(ii) If k contains (respectively, does not contain) μ_p, then any closed subgroup of infinite index (respectively, of arbitrary index) of $G_k^{(p)}$ is a free pro-p group.

(iii) G_k is topologically finitely generated.

Slimness and Elasticity

Theorem (Lubotzky-Melnikov-van den Dries, cf. [FJ] Proposition 24.10.3)

Let p be a prime number; F be a free pro-p group with $2 \leq \mathrm{rank}(F) \leq \omega$; N be a nontrivial closed normal subgroup of infinite index of F. Then N is a free pro-p group with $\mathrm{rank}(N) = \omega$.

Proposition 4.2

Let k be a p-adic local field; $\overline{k}$ be an algebraic closure of k; $G_k \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{k}/k)$ be the absolute Galois group of k. Then:

- G_k, as well as any almost pro-p-maximal quotient Q of G_k is slim.
- G_k, as well as any almost pro-p-maximal quotient Q of G_k is elastic.

Slimness and Elasticity

Proposition 4.3

(i) Let F be a NF [i.e. a number field] with an algebraic closure of $\overline{F}$; $\tilde{F}$ be a solvably closed field extension of F; put $G_F \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{F}/F)$ and $Q_F \stackrel{\text{def}}{=} \mathrm{Gal}(\tilde{F}/F)$. Then Q_F is slim and elastic.

(ii) In the situation of (i), let $v \neq w$ be non-archimedean valuations of F; G_v, G_w (resp. Q_v, Q_w) be the decomposition groups of v, w in G_F (resp. Q_F) [up to conjugation]. Then Q_v equals the image of $G_v \subseteq G_F$ in Q_F; G_v (resp. Q_v) is commensurably terminal in G_F (resp. Q_F); $G_v \subseteq G_F$ is relatively slim; $G_v \cap G_w = \{1\}$.

Slimness and Elasticity

Proposition 4.4

Any profinite group Δ of GFG-type is topologically finitely generated.

Proposition 4.5

(i) Let Δ be a profinite group of GFG-type that admits partial construction data (k, X, Σ), such that X is a hyperbolic orbicurve and Σ contains a prime invertible in k. Then Δ is slim and elastic.

(ii) Let $1 \to \Delta \to \Pi \to G \to 1$ be an extension of GSAFG-type that admits partial construction data (k, X, Σ), such that X is a hyperbolic orbicurve, $\Sigma \neq \varnothing$ and k is either an MLF or an NF. Then Π is slim, but not elastic.

Slimness and Elasticity

Definition 4.6

(i) We say a field k is **sub-p-adic**, if k is a subfield of a finitely generated field over Q_p for some prime number p.
(ii) We say a field k is **Kummer-faithful**, if k is of characteristic 0, and if for any finite extension k' of k and any semi-abelian variety A over k, we have $\bigcap_{N \geq 1} N \cdot A(k') = \{0\}$.
(iii) We say a field k is **l-cyclotomically full**, if the l-adic cyclotomic character $\chi_l : G_k \to \mathbb{Z}_l^{\times}$ has an open image.

Proposition 4.7

(i) k is sub-p-adic $\Rightarrow$ k is Kummer-faithful $\Rightarrow$ k is l-cyclotomically full for any l.
(ii) If k is Kummer-faithful, then any finitely generated field over k is also Kummer-faithful.
(iii) If k is Kummer-faithful, then G_k is slim.
(iv) If k is a Hilbertian field, then G_k is slim.

Slimness and Elasticity

Proposition (Minamide-Tsujimura)

Suppose that K is a Henselian discrete valuation field such that the residue field k of K is of characteristic p. Then G_K is slim and elastic. Moreover, the following hold:

- G_K is not topologically finitely generated if and only if k is infinite, or $\mathrm{char}(K) = p$.
- If k is infinite, and $\mu_p \subseteq K$ in the case where $\mathrm{char}(K) = 0$, then any almost pro-p-maximal quotient of G_K is slim, elastic, and not topologically finitely generated.
- If k is finite, then any almost pro-p-maximal quotient of G_K is slim and elastic.

The End

Thank you for your attention!