Cuspidalisations in Anabelian Geometry

Week 5: Decomposition groups

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I. Decomposition groups

Decomposition groups

Let k be a perfect field, X be a hyperbolic orbicurve over k with sommth compactification $\overline{X}$. Write X^{cl} (resp., $X^{\mathrm{cl}+}$) for the set of closed points of X (resp. of $\overline{X}$), and write $\mathrm{Cusp}(X)$ for the set of cusps of X.

For any $x \in X^{\mathrm{cl}+}$ and $Y \to X$ tamely finite étale, When X is scheme-like, let $\overline{Y}$ denote the desingularization of the normalization of $\overline{X}$ in Y. [When X is an orbicurve, then we can passing to a scheme-like Galois covering X' of X, and only consider about finite étale coverings of X'.]

Let $y \in Y^{\mathrm{cl}+}$ lying over x, then the decomposition group (resp. inertia group) of Y/X at y is defined as

$$D_y(Y/X) \overset{\mathrm{def}}{=} \{g \in \mathrm{Aut}_X(Y) : g(y) = y\};\ I_y(Y/X) \overset{\mathrm{def}}{=} \{g \in \mathrm{Aut}_X(Y) : g(a) = a, \forall a \in \kappa(y)\}.$$

Decomposition groups

Let $(X_i)_{i\in I}$ be a cofinal system in $\mathcal{B}^{\mathrm{tame}}(X)$, with compatibale cofinal system $(\overline{X}_i)_{i\in I}$. We shall call $\tilde{X} \stackrel{\mathrm{def}}{=} \varprojlim_{i\in I} X_i$ a pro-object of $\mathcal{B}^{\mathrm{tame}}(X)$ or a universal pro-covering of X, and we have

$$\pi_1^{\mathrm{tame}}(X) \cong \mathrm{Aut}_X(\tilde{X}) \stackrel{\mathrm{def}}{=} \varprojlim_{i\in I} \mathrm{Aut}_X(X_i).$$

Let $x \in X^{\mathrm{cl}+}$ be a closed point or a cusp of X. Choose a pro-point $\tilde{x} = (x_i)_{i\in I}$ [where $x_i \in X_i^{\mathrm{cl}+}$] lying over x.

Decomposition groups

Definition 1.1

The decomposition group (resp. inertia group) of $\tilde{X}$ at $\tilde{x}$ is defined as

$$D_{\tilde{x}} \stackrel{\text{def}}{=} \varprojlim_{i \in I} D_{x_i}(X_i/X); \quad I_{\tilde{x}} \stackrel{\text{def}}{=} \varprojlim_{i \in I} I_{x_i}(X_i/X).$$

The decomposition group D_x (resp. the inertia group I_x) of X at x is defined as $D_{\tilde{x}}$ (resp. $I_{\tilde{x}}$) for any choice of $\tilde{x}$, which is well-defined upto conjugacy of $\mathrm{Aut}_X(\tilde{X})$.

Decomposition groups

Example 1.2

Suppose that X is a hyperbolic curve and $\mathrm{char}(k) = 0$, then we have an exact sequence

$$1 \to I_x \to D_x \to \mathrm{Gal}(\overline{\kappa(x)}/\kappa(x)) \to 1.$$

Moreover, if $x \in X^{\mathrm{cl}}$, then $I_x = \{1\}$; if $x \in \mathrm{Cusp}(X)$, then $I_x \cong \widehat{\mathbb{Z}}(1)$.

Corollary 1.3

If $x \in X^{\mathrm{cl}+}$ and k is algebraically closed, then $I_x = D_x$.

Decomposition groups

Let $\Sigma \subseteq \mathfrak{Primes}$ be a nonempty set of primes such that $\text{char}(k) \notin \Sigma$, put $\Delta_X \stackrel{\text{def}}{=} \pi_1^{\text{tame}}(X)^{(\Sigma)}$. For any cusp x of X, write $I_x \subseteq \Delta_X$ for the image [up to conjugation] of the inertia group of x in Δ_X.

Proposition 1.4

(i) If X is of type (g, r), then Δ_X is isomorphic to the pro-Σ completion of the surface group of type (g, r).

(ii) X is non-proper [i.e., $r > 0$] iff Δ_X is free pro-Σ.

Decomposition groups

Proposition 1.5

The sub-group $I_x \subseteq \Delta_X$ is commensurably terminal.

Sketch of proof. Suppose that $g \in C_{\Delta_X}(I_{\tilde{x}}) \setminus I_{\tilde{x}}$, then we can choose a small enough open subgroup $\Delta_Y \subseteq \Delta_X$, such that Y has distinct cusps $y, y', g(y)$ mapping to x [and $\tilde{x}$ maps to y]. Then since $g \in C_{\Delta_X}(I_{\tilde{x}})$, we can choose a pro-$\Sigma$ finite étale covering $Z \to X$, which has genus ≥ 2 and cusps $z, z', g(z)$ lying over $y, y', g(y)$ respectively, such that $I_z = I_{g(z)}$. Then by the group structure of Δ_Z^{ab}, it is easy to see that Z admits an infinite abelian pro-Σ covering which is totally ramified at z, z' but not at $g(z)$, which contradicts to $I_z = I_{g(z)}$,

II. Decomposition Groups of Hyperbolic Orbicurves

Decomposition Groups of Hyperbolic Orbicurves

Let Σ be a nonempty set of prime numbers, Δ a pro-Σ group of GFG-type that admits base-prime [i.e. every prime dividing the order of a finite quotient group of Δ is invertible in k] partial construction data (k, X, Σ) such that X is a hyperbolic orbicurve, and k is algebraically closed.

Hence $\mathrm{char}(k) \notin \Sigma$; Δ is isomorphic to the maximal pro-Σ quotient Δ_X of $\pi_1^{\mathrm{tame}}(X_{\overline{k}})$; when X is not scheme-like, it admits a scheme-like finite étale Galois covering Y such that $\mathrm{Gal}(Y/X)$ is pro-Σ.

Let x_A (respectively, $x_B \neq x_A$) be either a closed point or a cusp [up to conjugation] of X; $A \subseteq \Delta$ (respectively, $B \subseteq \Delta$) [the image of] the decomposition group of x_A (respectively, x_B) in Δ [via the composition $D_{x_A}, D_{x_B} \hookrightarrow \pi_1^{\mathrm{tame}}(X) = \pi_1^{\mathrm{tame}}(X_{\overline{k}}) \twoheadrightarrow \Delta_X \xrightarrow{\sim} \Delta$].

Decomposition Groups of Hyperbolic Orbicurves

Proposition (Decomposition Groups of Hyperbolic Orbicurves)

(i) A, B are pro-cyclic groups; $A \cap B = \{1\}$. If x_A is a closed point of X, and $A \neq \{1\}$, then A is a finite, normally terminal subgroup of Δ. If x_A is a cusp, then A is a torsion-free, commensurably terminal infinite subgroup of Δ.

(ii) The order of every finite cyclic closed subgroup $C \subseteq \Delta$ divides the order of X.

(iii) Every finite nontrivial closed subgroup $C \subseteq \Delta$ is contained in a decomposition group of a unique closed point of X. In particular, the nontrivial decomposition groups of closed points of X may be characterized ["group-theoretically"] as the maximal finite nontrivial closed subgroups of Δ.

(iv) X is a hyperbolic curve if and only if Δ is torsion-free.

Proposition (Decomposition Groups of Hyperbolic Orbicurves)

(v) Suppose that the quotient $\psi_A : \Delta \to \Delta_A$ of Δ by the closed normal subgroup of Δ topologically generated by A is slim and nontrivial. If x_A is a closed point of X (resp., a cusp), then we suppose further that $\Sigma = \mathfrak{Primes}$ [which forces $\mathrm{char}(k) = 0$] (resp., that $A \subseteq J$ for some normal open torsion-free subgroup J of Δ).

Then Δ_A is a profinite group of GFG-type that admits base-prime partial construction data (k, X_A, Σ) such that X_A is a hyperbolic orbicurve equipped with a dominant k-morphism $\varphi_A : X \to X_A$ that is uniquely determined [up to a unique isomorphism] by the property that it induces [up to composition with an inner automorphism] ψ_A.

Decomposition Groups of Hyperbolic Orbicurves

Proposition (Decomposition Groups of Hyperbolic Orbicurves)

Moreover, if x_A is a closed point of X (resp., a cusp), then φ_A is a partial coarsification morphism which is an isomorphism either over X_A, or over the complement in X_A of the point of X_A determined by x_A (resp., is an open immersion whose image is the complement of the point of X_A determined by x_A).

(vi) In the notation of (v), if $B \neq \{1\}$, then $\psi_A(B) \neq \{1\}$.

III. Cuspidal Decomposition Groups — The Case of MLF's

Cuspidal Decomposition Groups — The Case of MLF's

Notations:

- k: a p-adic local field with absolute Galois group G_k.
- X: a hyperbolic curve of type (g, r) over k, where $2g - 2 + r > 0$.
- Σ: a non-empty set of primes, with contains at least one prime $\ell \neq p$.
- $\Delta_X \stackrel{\mathrm{def}}{=} \pi_1(X_{\overline{k}})^{(\Sigma)}$: the pro-$\Sigma$ geometric fundamental group of X.
- $\Pi_X \stackrel{\mathrm{def}}{=} \pi_1(X)/\operatorname{Ker}(\pi_1(X_{\overline{k}}) \twoheadrightarrow \Delta_X)$: the pro-$\Sigma$ arithmetic fundamental group of X.
- $1 \to \Delta_X \to \Pi_X \to G_k \to 1$: the pro-$\Sigma$ homotopy exact sequence of X.

Cuspidal Decomposition Groups — The Case of MLF's

Question

Can we reconstruct group-theoretically the cuspidal decomposition groups in Π_X from Σ and the pro-Σ homotopy exact sequence

$$1 \to \Delta_X \to \Pi_X \to G_k \to 1?$$

Cuspidal Decomposition Groups — The Case of MLF's

Proposition (Mono-anabelian Reconstruction for MLF's)

There exist functorial group-theoretic algorithms for constructing, from a group G of MLF-type, i.e. $G \cong G_k$,

- $p(G)$, $d(G)$, $e(G)$, $f(G)$, $I(G)$, $P(G)$ and $\mathrm{Frob}(G)$,

which "correspond" to

- $p(G_k) = p_k$ the residue characteristic of k; $d(G_k) = d_k \stackrel{\mathrm{def}}{=} [k : \mathbb{Q}_{p_k}]$; $e(G_k) = e_k$ the absolute ramification index of k: $f(G_k) = f_k$ the residue degree of $\kappa(k)$ over $\mathbb{F}_{p_k}$.
- $I(G_k) = I_k$ the inertia subgroup of G_k; $P(G_k) = P_k$ the wild inertia subgroup of G_k; $\mathrm{Frob}(G_k) = \mathrm{Frob}_k$ the Frobenius element in G_k/I_k.

Cuspidal Decomposition Groups — The Case of MLF's

Definition (Weight of repersentations of MLF's)

Let M be a $\mathbb{Q}_\ell$ vector space with continuous G_k-action where $\ell \neq p$, such that the action of the inertia subgroup I_k on M is quasi-unipotent.

For any $w \in \mathbb{Z}$, choose an arbitrary lifting of $\mathrm{Frob}_k \in G_k/I_k$ in G_k, and write $M^{\mathrm{wt}=w}$ for the $\mathbb{Q}_\ell$-subspace of M on which Frobenius acts with eigenvalues of weight w [i.e. algebraic numbers with absolute values $q^{w/2}$].

Note that the weight is independent of the choice of a lifting of the Frobenius element since the action of the inertia subgroup on Δ_X^{ab} is quasi-unipotent.

Example: The Tate module $\mathbb{Z}_\ell(1)$ is of weight 2.

Cuspidal Decomposition Groups — The Case of MLF's

Proposition (Reconstruction of the type of hyperbolic curves)

(i) X is non-proper [i.e., $r > 0$] if and only if Δ_X is a free pro-Σ group.

(ii) If $r > 0$, then

$$r = \dim_{\mathbb{Q}_\ell}\left(\Delta_X^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_\ell\right)^{\mathrm{wt}=2} - \dim_{\mathbb{Q}_\ell}\left(\Delta_X^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_\ell\right)^{\mathrm{wt}=0} + 1, \quad \text{for } \ell \in \Sigma,\ \ell \neq p,$$

and for $r \geq 0$,

$$g = \frac{1}{2}\left(\dim_{\mathbb{Q}_{p'}}\left(\Delta_X^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_{p'}\right) - \max\{0, r-1\}\right), \quad \text{for any } p' \in \Sigma.$$

Cuspidal Decomposition Groups — The Case of MLF's

Let X be an affine hyperbolic curve over k, and $\overline{X}$ the canonical smooth compactification. We shall fix a prime $\ell \in \Sigma$, $\ell \neq p$.

Proposition (Reconstruction of cuspidal decomposition subgroups)

(i) Reconstruction of the natural surjection $\Delta_X \twoheadrightarrow \Delta_{\overline{X}}$: An open subgroup $H \subseteq \Delta_X$ contains $\mathrm{Ker}(\Delta_X \twoheadrightarrow \Delta_{\overline{X}})$ if and only if $r(X_H) = [\Delta_X : H] \cdot r(X)$, where X_H is the covering corresponding to $H \subseteq \Delta_X$, and $r(-)$ denotes the number of cusps which can be computed group-theoretically.

Cuspidal Decomposition Groups — The Case of MLF's

Proposition (Reconstruction of cuspidal decomposition subgroups)

(ii) Reconstruction of pro-ℓ cuspidal inertia subgroups: A closed subgroup $A \subseteq \Delta_X^{(\ell)}$ isomorphic to $\mathbb{Z}_\ell$ is contained in the inertia subgroup of a cusp if and only if, for any open subgroup $\Delta_Y^{(\ell)} \subseteq \Delta_X^{(\ell)}$, the composite $A \cap \Delta_Y^{(\ell)} \hookrightarrow \Delta_Y^{(\ell)} \twoheadrightarrow (\Delta_{\overline{Y}}^{(\ell)})^{\mathrm{ab}}$ vanishes. Here, $\overline{Y}$ denotes the canonical smooth compactification of Y. Hence the inertia subgroup of a cusp is the maximal closed subgroup $A \subseteq \Delta_X^{(\ell)}$ isomorphic to $\mathbb{Z}_\ell$ with the above property.

Cuspidal Decomposition Groups — The Case of MLF's

Sketch of proof: The "only if" part is trivial.

For the "if" part, for any $\Delta_Y^{(\ell)}$, let $Y \to Z \to X$ be the coverings corresponding to $\Delta_Y^{(\ell)} \subseteq \Delta_Z^{(\ell)} \overset{\mathrm{def}}{=} A \cdot \Delta_Y^{(\ell)} \subseteq \Delta_X^{(\ell)}$. Then $\Delta_Z^{(\ell)}/\Delta_Y^{(\ell)} \cong A/(A \cap \Delta_Y^{(\ell)}) \cong \mathbb{Z}/\ell^N\mathbb{Z}$ for some N, which is abelian.

By the assumption for $\Delta_Z^{(\ell)}$, $A = A \cap \Delta_Z^{(\ell)}$ vanishes in $(\Delta_{\overline{Z}}^{(\ell)})^{\mathrm{ab}}$, hence the image of A in $(\Delta_Z^{(\ell)})^{\mathrm{ab}}$ is contained in the subgroup generated by the image of the inertia subgroups in $\Delta_Z^{(\ell)}$. Since A is pro-cyclic, the image of A is contained in the image of I_z for some cusp z of Z.

Then Since A surjects via $A \to (\Delta_Z^{(\ell)})^{\mathrm{ab}} \twoheadrightarrow \Delta_Z^{(\ell)}/\Delta_Y^{(\ell)} \cong A/(A \cap \Delta_Y^{(\ell)}) \cong \mathbb{Z}/\ell^N\mathbb{Z}$, we get a surjection $I_z \twoheadrightarrow \Delta_Z^{(\ell)}/\Delta_Y^{(\ell)}$, hence $Y \to Z$ is totally ramified at z. Then the "if" part follows.

Cuspidal Decomposition Groups — The Case of MLF's

Proposition (Reconstruction of cuspidal decomposition subgroups)

(iii) We can reconstruct the set of cusps of X as the set of $\Delta_X^{(\ell)}$-orbits of the inertia subgroups in $\Delta_X^{(\ell)}$ via conjugation.

(iv) By functorially reconstructing the cusps of any covering $Y \to X$ from $\Delta_Y \subseteq \Delta_X \subseteq \Pi_X$, we can reconstruct the set of cusps of the universal pro-covering $\widetilde{X} \to X$.

(v) We can reconstruct inertia subgroups in Δ_X as the subgroups that fix some cusp of the universal pro-covering $\widetilde{X} \to X$ of X determined by the basepoint under consideration.

(vi) We have a characterisation of decomposition groups D of cusps in Π_X as $D = C_{\Pi_X}(I)$ for some inertia subgroup I in Δ_X.

IV. Cuspidal Decomposition Groups — The General Case

Cuspidal Decomposition Groups — The General Case

Let G be a slim profinite group;

$$1 \to \Delta \to \Pi \to G \to 1$$

an extension of GSAFG-type that admits base-prime partial construction data $(k, \tilde{k}, X, \Sigma)$, where X is a hyperbolic orbicurve; $\alpha : \pi_1^{\mathrm{tame}}(X) \to \Pi$ a scheme-theoretic envelope; $\ell \in \Sigma$ a prime such that the cyclotomic character

$$\chi_G^{\mathrm{cyclo}} : G \to \mathbb{Z}_\ell^\times$$

[i.e., the character whose restriction to $\pi_1^{\mathrm{tame}}(X)$ via α and the surjection $\Pi \to G$ is the usual cyclotomic character $\pi_1^{\mathrm{tame}}(X) \to \mathrm{Gal}(\tilde{k}/k) \to \mathbb{Z}_\ell^\times$] has open image [i.e., is ℓ-cyclotomically full].

A character $\chi : G \to \mathbb{Z}_\ell^\times$ is called $\mathbb{Q}$-cyclotomic (of weight $w \in \mathbb{Q}$) if there exist integers $a, b > 0$ such that $\chi^b = (\chi_G^{\mathrm{cyclo}})^a$, $w = 2a/b$.

Cuspidal Decomposition Groups — The General Case

Propsition (Cuspidal Decomposition Groups)

(i) X is non-proper if and only if every torsion-free pro-Σ open subgroup of Δ is free pro-Σ.

(ii) Let M be a finite-dimensional $\mathbb{Q}_\ell$-vector space equipped with a continuous G-action. We say this action is quasi-trivial if it factors through a finite quotient of G. Let $\tau(M)$ denote the quasi-trivial rank of M, i.e., the sum of $\mathbb{Q}_\ell$-dimensions of the quasi-trivial subquotients M_j/M_{j+1} in any filtration

$$M_n \subseteq \cdots \subseteq M_j \subseteq \cdots \subseteq M_0 = M$$

by $\mathbb{Q}_\ell[G]$-modules such that each M_j/M_{j+1} is either quasi-trivial or has no nontrivial quasi-trivial subquotients.

Cuspidal Decomposition Groups — The General Case

Propsition (Cuspidal Decomposition Groups)

If $\chi : G \to \mathbb{Z}_\ell^\times$ is a character, define

$$d_\chi(M) := \tau(M(\chi^{-1})) - \tau(\mathrm{Hom}_{\mathbb{Q}_\ell}(M, \mathbb{Q}_\ell)),$$

where $M(\chi^{-1})$ denotes the twist of M by χ^{-1}. We say two characters $G \to \mathbb{Z}_\ell^\times$ are *power-equivalent* if some positive power of the two coincide. Then $d_\chi(M)$, regarded as a function of χ, depends only on the power-equivalence class of χ.

Cuspidal Decomposition Groups — The General Case

Propsition (Cuspidal Decomposition Groups)

(iii) Suppose X is not proper [cf. (i)]. Then the character $G \to \mathbb{Z}_\ell^\times$ arising from the determinant of the G-module $H^{\mathrm{ab}} \otimes \mathbb{Q}_\ell$, where $H \subseteq \Delta$ is a torsion-free pro-Σ characteristic open subgroup such that $H^{\mathrm{ab}} \otimes \mathbb{Q}_\ell \neq 0$, is $\mathbb{Q}$-cyclotomic of positive weight.

Moreover, for every sufficiently small characteristic open subgroup $H \subseteq \Delta$, the power-equivalence class of the cyclotomic character χ_G^{cyclo} may be characterized as the unique power-equivalence class of characters $\chi : G \to \mathbb{Z}_\ell^\times$ of the form $\chi = \chi^* \cdot \chi_*$, where χ^* (resp. χ_*) is a $\mathbb{Q}$-cyclotomic character $\chi_\bullet$ of maximal (resp. minimal) weight such that

$$\tau(M(\chi_\bullet^{-1})) \neq 0$$

for some subquotient G-module M of $(H^{\mathrm{ab}} \otimes \mathbb{Q}_\ell) \oplus \mathbb{Q}_\ell$ (where $\mathbb{Q}_\ell$ is equipped with trivial G-action).

In this situation, if $\chi = \chi_G^{\mathrm{cyclo}}$, then the divisor of cusps of the covering of $X \times_k \tilde{k}$ determined by H is a disjoint union of $d_\chi(H^{\mathrm{ab}} \otimes \mathbb{Q}_\ell) + 1$ copies of $\mathrm{Spec}(\tilde{k})$.

Cuspidal Decomposition Groups — The General Case

Propsition (Cuspidal Decomposition Groups)

(iv) Suppose X is not proper [cf. (i)]. Let $H \subseteq \Delta$ be a torsion-free pro-Σ characteristic open subgroup; $H \to H^*$ the maximal pro-l quotient of H. Then the decomposition groups of cusps in H^* may be characterized ["group-theoretically"] as the maximal closed subgroups $I \subseteq H^*$ isomorphic to $\mathbb{Z}_\ell$ such that the following holds:

$$d_{\chi_G^{\mathrm{cyclo}}}(J^{\mathrm{ab}} \otimes \mathbb{Q}_\ell) + 1 = [I \cdot J : J] \cdot d_{\chi_G^{\mathrm{cyclo}}}((I \cdot J)^{\mathrm{ab}} \otimes \mathbb{Q}_\ell) + 1,$$

[i.e., "the covering of curves corresponding to $J \subseteq I \cdot J$ is totally ramified at precisely one cusp"] for every characteristic open subgroup $J \subseteq H^*$.

Cuspidal Decomposition Groups — The General Case

Propsition (Cuspidal Decomposition Groups)

(v) Let X, H, H^* be as in (iv). Then the set of cusps of the covering of $X \times_k \tilde{k}$ determined by H is in natural bijective correspondence with the set of conjugacy classes in H^* of decomposition groups of cusps [as described in (iv)]. Moreover, this correspondence is functorial in H and compatible with the natural actions by Π on both sides. In particular, by allowing H to vary, this yields a ["group-theoretic"] characterization of the decomposition groups of cusps in Π.

(vi) Let $D \subseteq \Pi$ be a decomposition group of a cusp. Then $D = C_\Pi(D \cap \Delta)$.

The End

Thank you for your attention!