Cuspidalisations in Anabelian Geometry

Week 7: Applications of Chains

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2025

Review

What has been done

- Various group-theoretic properties arise from anabelian geometry.
- Group-theoretic characterisations of decomposition groups.
- The theory of $\widetilde{X}/X$-chains and Π-chains.

The set $\mathbb{D}$

Definition 1

We define the following sets:

- $\mathbb{V}$ is a set of isomorphism classes of algebraic stacks.
- $\mathbb{F}$ is a set of isomorphism classes of fields.
- $\mathbb{S}$ is a set of non-empty subsets of prime numbers.

We write $\mathbb{D} \subset \mathbb{V} \times \mathbb{F} \times \mathbb{S}$ for a subset of 3-tuples. We say that $\mathbb{D}$ is **chain-full** if every extension of GSAFG-type

$$1 \to \Delta \to \Pi \to G \to 1$$

for some slim G, that admits a base-prime partial construction data (X, k, Σ) such that $([X], [k], \Sigma) \in \mathbb{D}$, it holds that $([X_i], [k_i], \Sigma_i) \in \mathbb{D}$ for any X_i appeared as a member of an object in $\mathrm{Chain}(\widetilde{X}/X)$.

The set $\mathbb{D}$

Definition 2

Let $\mathbb{D}$ be a set as in Definition 1 such that $\mathbb{D}$ is chain-full. We say that **rel-isom-$\mathbb{D}$GC holds** if the followings hold: for $i \in \{1, 2\}$ and extensions of GSAFG-type

$$1 \to \Delta_i \to \Pi_i \to G_i \to 1$$

in $\mathbb{D}$ (in the sense of Definition 1), such that

$$\mathrm{Isom}_{k_1,k_2}(X_1, X_2) \xrightarrow{\sim} \mathrm{Isom}_{G_1,G_2}(\Pi_1, \Pi_2)/\mathrm{Inn}(\Delta_2)$$

is bijective.

The set $\mathbb{D}$

Definition 3

Let $\mathbb{D}$ be a set as in Definition 1 such that $\mathbb{D}$ is chain-full. We say that **rel-hom-$\mathbb{D}$GC holds** if the followings hold: for $i \in \{1, 2\}$ and extensions of GSAFG-type

$$1 \to \Delta_i \to \Pi_i \to G_i \to 1$$

in $\mathbb{D}$, such that

$$\mathrm{Hom}^{\mathrm{dom}}_{k_1,k_2}(X_1, X_2) \xrightarrow{\sim} \mathrm{Hom}^{\mathrm{open}}_{G_1,G_2}(\Pi_1, \Pi_2)/\mathrm{Inn}(\Delta_2)$$

is bijective.

The Main Theorem

Let $\mathbb{D}$ be a chain-full set such that rel-isom-$\mathbb{D}$GC holds true. For $i \in \{1, 2\}$, and let G_i be a slim profinite group. Moreover, let

$$1 \to \Delta_i \to \Pi_i \to G_i \to 1$$

be an extension of GSAFG-type that admits base-prime partial construction data $(k_i, \tilde{k}_i, X_i, \Sigma_i)$ such that $([X_i], [k_i], \Sigma_i) \in \mathbb{D}$, and $\alpha_i : \pi_1^{\mathrm{tame}}(X_i) \twoheadrightarrow \Pi_i$ is a scheme-theoretic envelope. Furthermore, assume that the following two conditions hold:
(a) If either X_1 or X_2 is a hyperbolic orbicurve, then both X_1 and X_2 are hyperbolic orbicurves.
(b) If either X_1 or X_2 is non-proper, then there exists a prime number $\ell \in \Sigma_1 \cap \Sigma_2$ such that for $i = 1, 2$, the cyclotomic character $G_i \to \mathbb{Z}_\ell^\times$ has open image.

The Main Theorem

Theorem 4

Let $\phi : \Pi_1 \xrightarrow{\sim} \Pi_2$ be an isomorphism of profinite groups such that ϕ restricts to an isomorphism $\phi_\Delta : \Delta_1 \xrightarrow{\sim} \Delta_2$, hence ϕ also induces an isomorphism $\phi_G : G_1 \xrightarrow{\sim} G_2$. Then
(i) The natural functors $\mathrm{Chain}(\widetilde{X}_i/X_i) \to \mathrm{Chain}(\Pi_i)$; $\mathrm{Chain}^{\text{iso-trm}}(\widetilde{X}_i/X_i) \to \mathrm{Chain}^{\text{iso-trm}}(\Pi_i)$; $\text{ÉtLoc}(\widetilde{X}_i/X_i) \to \text{ÉtLoc}(\Pi_i)$ are equivalences of categories that respect type-chains.
(ii) The isomorphism ϕ induces equivalences of categories: $\mathrm{Chain}(\Pi_1) \xrightarrow{\sim} \mathrm{Chain}(\Pi_2)$, $\mathrm{Chain}^{\text{iso-trm}}(\Pi_1) \xrightarrow{\sim} \mathrm{Chain}^{\text{iso-trm}}(\Pi_2)$ and $\text{ÉtLoc}(\Pi_1) \xrightarrow{\sim} \text{ÉtLoc}(\Pi_2)$. In particular, these equivalences respect type-chains and are functorial in ϕ.

The Main Theorem

Theorem 4 (continued)

(iii) Assume in addition, that rel-hom-$\mathbb{D}$GC holds true, and that for $i \in \{1, 2\}$, X_i is a hyperbolic orbicurve. Then the natural functors

$$\mathrm{Chain}^{\mathrm{trm}}(\widetilde{X}_i/X_i) \to \mathrm{Chain}^{\mathrm{trm}}(\Pi_i)\ ;\ \mathrm{DLoc}(\widetilde{X}_i/X_i) \to \mathrm{DLoc}(\Pi_i)$$

are equivalences of categories that respect type-chains.
(iv) Suppose that we are in the situation of assertion (iii), then the isomorphism ϕ induces equivalence of categories

$$\mathrm{Chain}^{\mathrm{trm}}(\Pi_1) \xrightarrow{\sim} \mathrm{Chain}(\Pi_2)\ ;\ \mathrm{DLoc}(\Pi_1) \xrightarrow{\sim} \mathrm{DLoc}(\Pi_2)$$

that respect type-chains and functorial in ϕ.

Example 5

Let p be a prime number, $\mathbb{S} \subset \mathfrak{Primes}$ be a set containing p. Let $\mathbb{V}$ be the isomorphism classes of hyperbolic orbicurves over fields with cardinality $\leq$ the cardinality of $\mathbb{Q}_p$.
(i) Let $\mathbb{F}$ be the set of isomorphism classes of generalised sub-p-adic fields (i.e. subfields of finitely generated extensions over $(\mathbb{Q}_p^{\text{unr}})^\wedge$). Let $\mathbb{D} := \mathbb{V} \times \mathbb{F} \times \mathbb{S}$. Then Theorem 4 (i) (ii) holds unconditionally in this case, i.e. $\mathbb{D}$ is chain-full and rel-isom-$\mathbb{D}$GC holds.
(ii) Let $\mathbb{F}$ be the set of isomorphism classes of sub-p-adic fields (i.e. subfields of finitely generated extensions over $\mathbb{Q}_p$) and let $\mathbb{D} := \mathbb{V} \times \mathbb{F} \times \mathbb{S}$. Then $\mathbb{D}$ is chain-full and rel-hom-$\mathbb{D}$GC holds. In particular, Theorem 4 (iii) and (iv) holds true unconditionally.

Examples

Definition 6

Let k be a field of characteristic 0. We say a k-scheme X is a hyperbolically fibered surface if there exists some hyperbolic curve X_0/k such that $X \to X_0$ admits the structure of a family of hyperbolic curves.

Definition 7

Let k be a field of characteristic 0. We say a generically scheme-like smooth geometrically connected algebraic stack X over k is an iso-poly-hyperbolic orbisurface if X admits a finite étale cover which is a hyperbolically fibered surface over some finite extension k'/k.

Example 8

Let p be a prime number, $\mathbb{S} := \mathfrak{Primes}$. Let $\mathbb{F}$ be the set of isomorphism classes of sub-p-adic fields and let $\mathbb{V}$ be the set of isomorphism classes of iso-poly-hyperbolic orbisurfaces over sub-p-adic fields. Let $\mathbb{D} := \mathbb{V} \times \mathbb{F} \times \mathbb{S}$. Then Theorem 4 (i),(ii) holds unconditionally.

Proof of Theorem 4

Proof of (i)

Consider the natural functor

$$\mathcal{F} : \mathrm{Chain}(\widetilde{X}_i/X_i) \to \mathrm{Chain}(\Pi_i)$$

$$X_i \mapsto \Pi_i.$$

To verify that this functor is an equivalence of categories, it suffices to verify that this natural functor respect elementary operations $\rightsquigarrow \in \{\curlywedge, \curlyvee, \bullet, \circledcirc\}$, i.e.
$\rightsquigarrow (X_i, X_{i+1}) \xrightarrow{\sim} \rightsquigarrow (\Pi_i, \Pi_{i+1})$ is bijective.

Proof of Theorem 4

Proof of (i)

Case 1: $\rightsquigarrow = \curlywedge$. The injectivity of the natural map

$$\curlywedge(X_i, X_{i+1}) \to \curlywedge(\Pi_i, \Pi_{i+1})$$

is clear. The surjectivity of this natural map follows from the definition of $\Pi_{(-)}$.
Case 2: $\rightsquigarrow = \curlyvee$. The injectivity of the natural map

$$\curlyvee(X_i, X_{i+1}) \to \curlyvee(\Pi_i, \Pi_{i+1})$$

is clear. The surjectivity of this natural map follows from the assumption that rel-isom-$\mathbb{D}$GC holds true.

Proof of Theorem 4

Proof of (i)

Case 3: $\rightsquigarrow = \bullet$. The injectivity of the natural map

$$\bullet(X_i, X_{i+1}) \to \bullet(\Pi_i, \Pi_{i+1})$$

is clear. Now let $\mathrm{pr}_i : \Pi_i \to \Pi_{i+1}$ be the surjective morphism determined by $\bullet$. Then it follows from Section 2 of Zhongpeng's 2nd talk (c.f. Week 5), that pr_i must arises uniquely from an open immersion $X_i \hookrightarrow X_{i+1}$.

Case 4: $\rightsquigarrow = \circledcirc$. The injectivity of the natural map

$$\circledcirc(X_i, X_{i+1}) \to \circledcirc(\Pi_i, \Pi_{i+1})$$

is clear. Let $\mathrm{pr}_i : \Pi_i \to \Pi_{i+1}$ be the surjective morphism determined by $\circledcirc$. It follows from Section 2 of Zhongpeng's 2nd talk, that pr_i must arises uniquely from a partial coarsification morphism $X_i \to X_{i+1}$.

Proof of Theorem 4

Proof of (ii)

Consider the isomorphism $\phi : \Pi_1 \xrightarrow{\sim} \Pi_2$, clearly it induces a natural functor

$$\mathcal{G}_\phi : \mathsf{Chain}(\Pi_1) \to \mathsf{Chain}(\Pi_2)$$

$$(\Pi_1)_j \mapsto (\phi(\Pi_1))_j.$$

To prove that $\mathcal{G}_\phi$ is an equivalence, it suffices to verify that $\mathcal{G}_\phi$ respect elementary operations $\rightsquigarrow \in \{\curlywedge, \curlyvee, \bullet, \circledcirc\}$, i.e. $\rightsquigarrow ((\Pi_1)_j, (\Pi_1)_{j+1}) \to \rightsquigarrow ((\Pi_2)_j, (\Pi_2)_{j+1})$. But it follows from Sean's talk (c.f. Definition of Chain(Π)) together with Section IV in Zhongpeng's 2nd talk, that elementary operations are group-theoretic. Hence the operations are preserved by isomorphisms hence $\mathcal{G}_\phi$ respect elementary operations.

The End