Cuspidalisations in Anabelian Geometry

Week 8: Elliptic Cuspidalizations

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I. Elliptically admissibility

Arithmeticity and coricity

Reference: [CanLift] Mochizuki, The Absolute Anabelian Geometry of Canonical Curves.

Let X be a hyperbolic orbicurve over a field k. Denote by

$$\overline{\mathrm{Loc}}(X)$$

the category whose objects are finite étale quotients of finite étale coverings of X, and whose morphisms are finite étale morphisms of algebraic stacks. Denote by

$$\overline{\mathrm{Loc}}_k(X)$$

the categorie obtained as above, except that we assume all the morphisms to be k-morphisms.

Arithmeticity and coricity

Definition ([CanLift], §2)

(i) X will be called arithmetic if $\overline{\mathrm{Loc}}(X)$ does not admit a terminal object.
(ii) X will be said to admit a(n) (absolute) core if there exists a terminal object Z in $\overline{\mathrm{Loc}}(X)$. In this case, $\overline{\mathrm{Loc}}(X) = \overline{\mathrm{Loc}}(Z)$, so we shall say that Z is a core.
(iii) X will be called k-arithmetic if $\overline{\mathrm{Loc}}_k(X)$ does not admit a terminal object.
(iv) X will be said to admit a k-core if there exists a terminal object Z in $\overline{\mathrm{Loc}}_k(X)$. In this case, $\overline{\mathrm{Loc}}_k(X) = \overline{\mathrm{Loc}}_k(Z)$, so we shall say that Z is a k-core.

Example (Sijsling, "Canonical models of arithmetic (1; e)-curves")

Let $X = E \setminus \{O\}$ be a punctured elliptic curve over a field k of characteristic zero. If $j(E) \notin \{0, 2^6 \cdot 3^3, 2^2 \cdot 73^3 \cdot 3^{-4}, 2^{14} \cdot 31^3 \cdot 5^{-3}\}$, then X admits a k-core $C \stackrel{\mathrm{def}}{=} X/\{\pm 1\}$.

Set up

- Let X be a hyperbolic orbicurve over a field k of characteristic zero; $\overline{k}$ an algebraic closure of k.
- We shall denote the base-change operation "$\times_k \overline{k}$" by means of a subscript $\overline{k}$.
- We have an exact sequence $1 \to \pi_1(X_{\overline{k}}) \to \pi_1(X) \to \mathrm{Gal}(\overline{k}/k) \to 1$.
- Let $\pi_1(X) \to \Pi$ be a quotient of profinite groups, and write $\Delta \subseteq \Pi$ for the image of $\Pi_1(X_{\overline{k}})$ in Π.
- Let $\Sigma = \Sigma(\Delta)$ for the smallest set of primes such that some open subgroup of Δ is pro-Σ.

Elliptically admissibility

Definition 3.1

Let $\pi_1(X) \to \Pi$ be a quotient of profinite groups defined as before. Then we shall say that X is Π-elliptically admissible if the following conditions hold:

(a) X admits a k-core $X \to C$ and a finite étale covering $Y \to X$.

(b) C is semi-elliptic, which admits a double covering $D \to C$ by a once-punctured elliptic curve D.

(c) Y is defined over a finite extension k_Y of k; Y arises from a normal open subgroup $\Pi_Y \subseteq \Pi$; $Y \to C$ factors as the composite of a covering $Y \to D$ with the covering $D \to C$

(d) Δ is pro-Σ, and moreover, the degree of the covering $Y \to C \times_k k_Y$ is a Σ integer.

When $\Pi = \Pi_1(X)$, we shall simply say that X is elliptically admissible.

Scheme-theoretic Elliptic Cuspidalizations

Let $N \geq 1$, E be an elliptic curve defined over a finite Galois extension k' of k, such that the N-torsion points $E[N]$ of $E_{\overline{k}}$ are defined over k'. Suppose that $\mathrm{Gal}(\overline{k}/k)$ is slim. Define

$$D \stackrel{\mathrm{def}}{=} E \setminus O, \quad C \stackrel{\mathrm{def}}{=} D/\{\pm 1\}, \quad U \stackrel{\mathrm{def}}{=} E \setminus E[N].$$

Then we have a finite étale covering $[N]_D : U \to D$ of degree N^2 which is induced by the multiplication by N map $[N]_E : E \to E$. Consider the chain

$$D \rightsquigarrow U \stackrel{\mathrm{def}}{=} U_{N^2} \rightsquigarrow U_n \rightsquigarrow U_{n-1} \rightsquigarrow \cdots \rightsquigarrow U_2 \rightsquigarrow U_1 \stackrel{\mathrm{def}}{=} D,$$

where $n \stackrel{\mathrm{def}}{=} N^2 - 1$, and U_k is a hyperbolic curve of type $(1, k)$ for each $1 \leq k \leq N^2$. The associated type-chain is

$$\curlywedge, \overbrace{\bullet, \ldots, \bullet}^{n}.$$

The above chain may thought of as a construction of a "cuspidalization" $U \to D$ of D.

Scheme-theoretic Elliptic Cuspidalizations

Now let X be an elliptically admissible hyperbolic orbicurve over k, and that we have been given finite étale coverings $V \to X$, $V \to D$ where V is defined over k'.

For simplicity, we shall **suppose** that $V \to X$ is a Galois covering such that $Gal(V/X)$ preserves the open subscheme $U_V \stackrel{\mathrm{def}}{=} V \times_D U \subseteq V$. Thus, $U_V \subseteq V$ descends to an open subscheme $U_X \subseteq X$.

Then we can obtain the chain

$$X \rightsquigarrow V \rightsquigarrow D \rightsquigarrow U \stackrel{\mathrm{def}}{=} U_{N^2} \rightsquigarrow U_n \rightsquigarrow U_{n-1} \rightsquigarrow \cdots \rightsquigarrow U_2 \rightsquigarrow U_1 \stackrel{\mathrm{def}}{=} D \rightsquigarrow V \rightsquigarrow X_* \stackrel{\mathrm{def}}{=} X,$$

whose associated type-chain is

$$\curlywedge, \curlyvee, \curlywedge, \overbrace{\bullet, \ldots, \bullet}^{n}, \curlywedge, \curlyvee.$$

The above chain may thought of as a construction of a "cuspidalization" $U_V \to V$ of V via the construction of "cuspidalization" $U \to D$ of D, equipped with descent data [i.e., a suitable collection of automorphisms] with respect to the finite étale Galois covering $V \to X$.

II. Elliptic Cuspidalization I: Algorithms

Set up

- Let $\mathbb{D}$ be a chain-full set of collections of partial construction data, such that the rel-isom-$\mathbb{D}$GC holds.
- Let G be a slim profinite group; $1 \to \Delta \to \Pi \to G \to 1$ an extension of GSAFG-type that admits partial construction data (k, X, Σ).
- Suppose that k is of characteristic zero; X is a Π-elliptically admissible hyperbolic orbicurve with $\alpha : \pi_1(X) \twoheadrightarrow \Pi$ the corresponding scheme-theoretic envelope; $([X], [k], \Sigma) \in \mathbb{D}$.
- Let $\widetilde{X} \to X$ be the pro-finite étale covering of X determined by α [so $\Pi \xrightarrow{\sim} \mathrm{Gal}(\widetilde{X}/X)$]; $\widetilde{k}$ the resulting field extension of k [so $G \xrightarrow{\sim} \mathrm{Gal}(\widetilde{k}/k)$].
- Suppose further that, for some $\ell \in \Sigma$, the cyclotomic character $G \to \mathbb{Z}_\ell^\times$ has open image, then by [AbsTopI], §4, the associated categories

$$\mathrm{Chain}(\Pi); \quad \mathrm{Chain}^{\text{iso-trm}}(\Pi); \quad \text{ÉtLoc}(\Pi)$$

can be constructed via purely "group-theoretic" operations from the extension $1 \to \Delta \to \Pi \to G \to 1$.

Elliptic Cuspidalizations

Let $N \geq 2$ be a Σ-integer [i.e. all the prime divisors of N belong to Σ], we shall recover the cuspidalization $\Pi_{U_X} \twoheadrightarrow \Pi$ from the extension $1 \to \Delta \to \Pi \to G \to 1$ and N.

Fix a sufficiently small open normal subgroup G' of G, which corresponds to a finite Galois extension k' over k. The choice of it would not change the final construction $\Pi_{U_X} \twoheadrightarrow \Pi$.

In §1, we have introduced scheme-theoretic elliptic cuspidalization of X. Let C be the k'-core of $X_{k'}$, and let $U \subseteq D, U_V \subseteq V, U_X \subseteq X$ be defined as in §1. Recall that G should be slim, and $\mathrm{Gal}(V/X) \cong \Pi/\Pi_V$ should preserve $U_V \subseteq V$, hence descends to $U_X \subseteq X$. In addition, U_X is independent to V by the coricity of C. We can recover the related profinite groups [along $\alpha : \pi_1(X) \twoheadrightarrow \Pi$]:

We can recover Π' by $\Pi' \overset{\mathrm{def}}{=} \Pi \times_G G'$.

Elliptic Cuspidalizations

We can recover $\Pi' \hookrightarrow \Pi_C$ [up to isomorphism] by the condition that C is a k'-core of $X_{k'}$.

The collection of $\Pi_D \subseteq \Pi_C$ that arise from double covering $D \to C$ [that exhibit C as semi-elliptic] can be recovered as the collection of open subgroups $J \subseteq \Pi_C$ of index 2, such that $J \cap \Delta_C$ is torsion-free [i.e., the covering determined by J is a scheme].

Fix $U \subseteq D$, and fix a sufficiently small open normal subgroup Π_V of Π_C, which corresponds to finite Galois coverings $V \to X, V \to D$. **Suppose** that $\mathrm{Gal}(V/X) \cong \Pi/\Pi_V$ preserves $U_V \subseteq V$. Moreover, we have extensions of GSAFG-type [which has not been recovered till now]

$$1 \to \Delta_U \to \Pi_U \to G' \to 1;\ 1 \to \Delta_{U_v} \to \Pi_{U_V} \to G' \to 1;\ 1 \to \Delta_{U_X} \to \Pi_{U_X} \to G \to 1.$$

Algorithms

(a) There exists a [not necessarily unique] Π-chain of the form

$$\Pi \rightsquigarrow \Pi_V \rightsquigarrow \Pi_D \rightsquigarrow \square_1 \stackrel{\text{def}}{=} \Pi_U \rightsquigarrow \cdots \rightsquigarrow \square_{N^2-1} \rightsquigarrow \Pi_D \rightsquigarrow \Pi_V \rightsquigarrow \Pi,$$

with associated type-chain

$$\curlywedge, \curlyvee, \curlywedge, \overbrace{\bullet, \ldots, \bullet}^{n=N^2-1}, \curlywedge, \curlyvee,$$

Hence the natural surjection $\Pi_U \twoheadrightarrow \Pi_D$ may be recovered from the chain of "$\bullet$'s" from $\square_1$ to the second Π_D.

Note: This means that the construction depends on the choice of [U and] D, and the assumption of the existense of V such that $\mathrm{Gal}(V/X) \cong \Pi/\Pi_V$ preserves $U_V \subseteq V$. The group Π_U can be characterized group-theoretic [up to isomorphism].

Algorithms

(b) We can recover $\Pi_{U_V} \twoheadrightarrow \Pi_V$ by pulling back $\Pi_U \twoheadrightarrow \Pi_D$ via $\Pi_V \hookrightarrow \Pi_D$.
Consider the outer action of Π/Π_V on Π_V. Since $\mathrm{Gal}(V/X) \xrightarrow{\sim} \Pi/\Pi_V$ preserves $U_V \subseteq V$, we also have an outer action of Π/Π_V on Π_{U_V}. Consider about the following diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_{U_V} & \longrightarrow & \Pi_{U_X} & \longrightarrow & \Pi/\Pi_V & \longrightarrow & 1 \\ & & \downarrow \wr & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Inn}(\Pi_{U_V}) & \longrightarrow & \mathrm{Aut}(\Pi_{U_V}) & \longrightarrow & \mathrm{Out}(\Pi_{U_V}) & \longrightarrow & 1 \end{array}$$

We can recover $\Pi_{U_X} \twoheadrightarrow \Pi$ by

$$\Pi_{U_X} \stackrel{\mathrm{def}}{=} \mathrm{Aut}(\Pi_{U_V}) \times_{\mathrm{Out}(\Pi_{U_V})} \Pi/\Pi_V \twoheadrightarrow \mathrm{Aut}(\Pi_V) \times_{\mathrm{Out}(\Pi_V)} \Pi/\Pi_V \xrightarrow{\sim} \Pi.$$

(c) Finally, the decomposition groups of the closed points of X lying in the complement of U_X may be obtained as the images via $\Pi_{U_X} \twoheadrightarrow \Pi_X$ of the cuspidal decomposition groups of Π_{U_X}.

III. Elliptic Cuspidalization II: Comparison

Set up

- Let $\mathbb{D}$ be a chain-full set of collections of partial construction data, such that the rel-isom-$\mathbb{D}$GC holds.
- For $i \in \{1, 2, \}$, let G_i be a slim profinite group; $1 \to \Delta_i \to \Pi_i \to G_i \to 1$ an extension of GSAFG-type that admits partial construction data (k_i, X_i, Σ_i).
- For $i \in \{1, 2, \}$, suppose that k_i is of characteristic zero; X_i is a Π_i-elliptically admissible hyperbolic orbicurve with $\alpha_i : \pi_1(X_i) \twoheadrightarrow \Pi_i$ the corresponding scheme-theoretic envelope; $([X_i], [k_i], \Sigma_i) \in \mathbb{D}$.
- Suppose further that, for some $\ell \in \Sigma_1 \cap \Sigma_2$, the cyclotomic character $G \to \mathbb{Z}_\ell^\times$ has open image.
- For $i \in \{1, 2, \}$, let $\Pi_{U_{X_i}} \twoheadrightarrow \Pi_i$ be the elliptic cuspidalization constructed in §2.

Comparison

Corollary 3.4

Let $\phi : \Pi_1 \to \Pi_2$ be an isomorphism of profinite groups such that $\phi(\Delta_1) = \Delta_2$.

Then there exists an isomorphism of profinite groups

$$\phi_U : \Pi_{U_{X_1}} \to \Pi_{U_{X_2}}$$

that is compatible with ϕ, relative to the natural surjections $\Pi_{U_{X_i}} \twoheadrightarrow \Pi_i$.

Moreover, such an isomorphism is unique up to composition with an inner automorphism arising from an element of the kernel of $\Pi_{U_{X_i}} \twoheadrightarrow \Pi_i$.

The End

Thank you for your attention!