Cuspidalisations in Anabelian Geometry

Week 9: Belyi Cuspidalisation

Yu Mao

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Preliminary

Definition 1

Let X_1, X_2 be hyperbolic orbicurves defined over fields k_1, k_2 respectively. We say that X_1 and X_2 are **isogenous** if there exists a hyperbolic orbicurve X over a field k, together with finite étale morphisms $X \to X_1$ and $X \to X_2$.

Preliminary

Definition 2

Let X be a hyperbolic orbicurve over a field k of characteristic 0. Then we say that X is of **strictly Belyi type** if X can be defined over a number field and X is isogenous to a hyperbolic curve of genus 0.

Scheme-theoretic Belyi Cuspidalisation for Tripods

Let k be a field of characteristic 0, let $P := \mathbb{P}^1 \setminus \{0, 1, \infty\}$ be defined over some finite Galois extension k'/k.
Let V be a hyperbolic curve over k' and let $U \subset V$ be an open subcurve. Assume that U is defined over a number field.
By Belyi's theorem, there exists a finite étale morphism

$$\beta : W \to P$$

for some open subcurve $W \subset U$.

Scheme-theoretic Belyi Cuspidalisation for Tripods

In particular, we have the following diagram:

$$\begin{array}{ccccc} W & \hookrightarrow & U & \hookrightarrow & V \\ \big\downarrow{\scriptstyle \beta} & & & & \\ P & & & & \end{array}$$

Furthermore, without the loss of generalities, we assume that cusps of W are defined over k' and that G_k is slim.

Scheme-theoretic Belyi Cuspidalisation for Tripods

The diagram determines the following chain for some non-negative integer m, n:

$$P \rightsquigarrow W \rightsquigarrow W_n \rightsquigarrow W_{n-1} \cdots \rightsquigarrow W_1 := U \rightsquigarrow U_m \rightsquigarrow \cdots \rightsquigarrow U_1 := V$$

whose type-chain is:

$$\curlywedge, \bullet, \ldots, \bullet.$$

Scheme-theoretic Belyi Cuspidalisation

Let X be a hyperbolic orbicurve of strictly Belyi type over k. Let $Q \subset P$ be an open subcurve. We write V for the common fintie étale cover of X and Q.
Assume for simplicity, that $V \to X$ is Galois and an open subscheme $U \subset V$ descends to an open subscheme $U_X \subset X$. Moreover, we assume that all cusps of Q are defined over k'. In particular, we have the following diagram

$$\begin{array}{ccccc} W & \hookrightarrow & U & \hookrightarrow & V \\ & & & \swarrow & \downarrow \\ \downarrow \beta & & Q & & X \\ & \swarrow & & & \\ P. & & & & \end{array}$$

Scheme-theoretic Belyi Cuspidalisation

In particular, we obtain the following chain for some non-negative integer m, n:

$$X \rightsquigarrow V \rightsquigarrow Q \rightsquigarrow Q_l \rightsquigarrow \dots Q_1 := P \rightsquigarrow W \rightsquigarrow W_n \rightsquigarrow \dots W_1 := U \rightsquigarrow U_m \cdots \rightsquigarrow U_1 := V \rightsquigarrow X_*$$

whose type-chain is:

$$\curlywedge, \curlyvee, \bullet, \dots, \bullet, \curlywedge, \bullet, \dots, \bullet, \curlyvee$$

together with a terminal isomorphism $X_* \xrightarrow{\sim} X$.

Belyi Cuspidalisation of Fundamental Groups

Let $\mathbb{D}$ be a chain-full set of partial construction data such that rel-isom-$\mathbb{D}$GC holds true. Let G be a slim profinite group and

$$1 \to \Delta \to \Pi \to G \to 1$$

be an extension of GSAFG type that admits partial construction data (k, X, Σ), where k is a field of characteristic 0, X is a hyperbolic curve of strictly Belyi type and $\Sigma := \mathfrak{Primes}$ is the set of all prime numbers such that $([k], [X], \Sigma) \in \mathbb{D}$. We write $\alpha : \Pi_X \to \Pi$ for the scheme-theoretic envelope. Suppose further that the ℓ-adic cyclotomic characters

$$\rho_\ell : G \to \mathbb{Z}_\ell^\times$$

has open image for every $\ell \in \Sigma$.

Belyi Cuspidalisation of Fundamental Groups

Theorem 2

For every open subcurve $U_X \subset X$ which can be defined over a number field, the natural surjection

$$\Pi_{U_X} \twoheadrightarrow \Pi_X$$

can be group-theoretically recovered from $1 \to \Delta \to \Pi \to G \to 1$. That is, there is a group-theoretic reconstruction of the surjection

$$\Pi^+ \twoheadrightarrow \Pi$$

such that the following diagram commutes

$$\begin{array}{ccc} \Pi_{U_X} & \twoheadrightarrow & \Pi_X \\ \downarrow{\scriptstyle \alpha^+} & & \downarrow{\scriptstyle \alpha} \\ \Pi^+ & \twoheadrightarrow & \Pi. \end{array}$$

Proof of Theorem 2

Step 1: Since X is of strictly Belyi type, there exists some open normal subgroup $\Pi_V \subset \Pi$, corresponds to a finite étale cover $V \to X$ such that there exists(not necessarily unique) a Π-chain whose type-chain is

$$\curlywedge, \curlyvee, \bullet, \ldots, \bullet, \curlywedge, \bullet, \ldots, \bullet, \curlyvee$$

that if we write $U := U_X \times_X V$ and $\Pi_U := \Pi_V \times_\Pi \Pi^+$, the profinite group Π_U together with the surjection $\Pi_U \twoheadrightarrow \Pi_V$ can be group-theoretically recovered from the type-chain above.

Belyi Cuspidalisation of Fundamental Groups

Proof of Theorem 2

Step 2: Consider the following commutative diagram with exact rows induced by the outer action of $\Gamma := \Pi/\Pi_V$ on Π_V:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_V & \longrightarrow & \Pi & \longrightarrow & \Gamma & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Pi_V & \longrightarrow & \mathrm{Aut}(\Pi_V) & \longrightarrow & \mathrm{Out}(\Pi_V) & \longrightarrow & 1. \end{array}$$

Proof of Theorem 2

Step 2 (continued): Recall that $\Pi_U \twoheadrightarrow \Pi_V$ is obtained from the elementary operation $\bullet$, which arises from geometry. Together with the slimness of Π, Π_V, Π_U, the outer action of Γ on Π_V lifts uniquely to Π_U. Hence we obtain the following exact sequence

$$1 \to \Pi_U \to \Pi_U \rtimes^{\mathrm{out}} \Gamma \to \Gamma \to 1$$

and we shall write $\Pi^+ := \Pi_U \rtimes^{\mathrm{out}} \Gamma$. Hence we have recovered the surjection

$$\Pi^+ \twoheadrightarrow \Pi.$$

Proof of Theorem 2

Step 3: By Zhongpeng's 2nd talk, we can group-theoretically recover the cuspidal decomposition groups of Π^+. In particular,

$$\alpha^+ : \Pi_{U_X} \to \Pi^+$$

is the scheme-theoretic envelope arising from α. $\square$

The End